

# Busy period, time of the first loss of a customer and the number of customers in $M^\infty|G^\delta|1|B$

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## Abstract

A two-sided exit problem is solved for a difference of a compound Poisson process and a compound renewal process. More precisely, the Laplace transforms of the joint distribution of the first exit time, the value of the overshoot and the value of a linear component at this instant are found. Further, we study the process reflected in its supremum. We determine the main two-boundary characteristics of the process reflected in its supremum. These results are then applied for studying the  $M^\infty|G^\delta|1|B$  system. We derive the distribution of a busy period and the numbers of customers in the system in transient and stationary regimes. The advantage is that these results are in a closed form, in terms of resolvent sequences of the process.

**Introduction** Queueing systems with batch arrivals and finite buffer have wide applications in the performance evaluation, telecommunications and manufacturing systems. One of the crucial performance issues of the single-server queue with finite buffer room is losses, namely, customers (packets, cells, jobs) that are not allowed to enter the system due to the buffer overflow. This issue is especially important in the analysis of telecommunication networks. Motivated by this fact, we derived the most important performance measures of queueing systems of this type. More precisely, we considered the  $M^\infty|G^\delta|1|B$  queueing system with finite buffer and its modification. We consider partial rejection, meaning that if an overflow of buffer occurs due to the arrival of a batch of customers, the amount of work brought by this batch is only partially admitted to the buffer, up to the limit of the free buffer space just before the arrival. The rest is rejected and therefore, is lost.

Evolution of the number of customers in such systems is described by a process with two reflecting boundaries. In general case this process is a difference of two

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compound renewal processes. Reflections from the upper boundary are generated by the supremum (infimum) of the process. Reflections from the lower boundary govern the server's behavior. In general, such processes are not Markovians, but by adding a complementary linear component (in some literature called age process), we obtain a Markov process, which describes functioning of the queueing system. Studying main characteristics of the system results to the investigating the two-boundary functionals of the governing process. We applied the solutions of the two-sided exit problem for the governing process (see [25] and [29] for the methodology) to obtain performance measures of interest. For the queueing system of  $M^\times|G^\delta|1|B$ ,  $G^\delta|M^\times|1|B$  (see[28]) type the governing process is the difference of the compound Poisson process and the compound renewal process complemented with the age process. The main result of this paper is the closed form formulae for the Laplace transforms of the busy period, time of the first loss and the number of customers in the system at arbitrary time.

First passage times of the level by Lévy processes in context of queues were considered in [12], where the explicit characterization of the Laplace transform of the busy period distribution was found for a finite capacity  $M|G|1$  queue, see also [44], [11] and [6]. In regard to finding the buffer overflow time we refer to [2], where arrivals are modeled by a Markov modulated Poisson process (MMPP) and service time is exponential, and also to [11], where  $BMAP|G|1|b$  queue was considered. Previous works on the overflow period were concentrated on simple Poisson arrivals [7], [10], or renewal arrivals [14] [42]. Picard and Lefèvre [45] found the distribution of the first crossing time of a Poisson process and renewal process in terms of polynomials of Abel-Gontcharoff types.

In recent years there has been a great interest in analyzing various queueing models with MAP (Markov Arrival Process) as input process or MSP (Markov service process). MAP is used to represent correlated traffic arising in modern telecommunication networks. In systems with Markov arrival or service processes (MAP, BMAP, or BMSP) and their modifications, it is common to use the supplementary variable methods and/or embedded Markov chains. For the method of supplementary variable we refer, for instance, to [15], where  $GI|MSP|1$  queue with finite as well as infinite buffer was analyzed. Embedded Markov chains techniques were used by [17]. De Boer et.al. [7] studied stationary distribution of the remaining service time upon reaching some target level in an  $M|G|1$  system. The asymptotic analysis of  $G|MSP|1|r$  queue has been carried out by [8]. Banik et.al. [4] found steady state distribution for a finite-buffer single-server queue  $GI|BMSP|1|N$  with renewal input. Random size batch service queueing models were subject of study in [13], [9] (stationary analysis of  $GI|BMSP|1$  queue) and [30] (asymptotic behavior of the loss probability).

Hence, the majority of recent literature is devoted mainly to the queue size and workload, most of the times in the steady state case. However, recently it was shown that steady-state parameters do not reflect the reality. A detailed discussion of the drawbacks of steady-state parameters in telecommunication networks may be found in [48]. This remark emphasizes importance of studying main performance measures in transient regime, which is the topic of this article. The main two-boundary characteristic of the governing process is the joint distribu-

tion of  $\{\chi, L, T\}$ , i.e. of the first exit time from the interval, the value of the overshoot and the value of the linear component at this instant. For the overview of the existing results on the two-boundary problems we refer to [25]. And here we only mention several authors who contributed a lot in the development of this area. Starting from Kemperman (1963), Takacs (1966), Emery [19], Pecherskii [43], Suprun, Shurenkov ([50], [51]), Lambert [35], Doney [16], Avram, Kyprianou, Pistorious [3], Pistorious [46], Kyprianou, Palmowsky [34], and Kadankov, Kadankova ([29], [27]) studied one- and two-boundary characteristics for different classes of stochastic processes.

The Laplace transforms of the joint distribution of the first exit time and the value of the overshoot at this time instant for general Lévy processes and random walks have been determined in [29]. The Laplace transforms of this joint distribution were found in terms of the Laplace transforms of the one-boundary characteristics of the process. This method for Lévy processes and random walks [29] was then applied for other classes of stochastic processes, such as the difference of compound renewal processes [27], and semi-Markov random walks with linear drift [24].

The rest of the article is structured as follows. In Section 1 we introduce the process, necessary notation and consider the one-boundary characteristics of the process. The two-sided problem is solved in Section 2. In this section we also prove the weak convergence of the joint distribution of the supremum, infimum and the value of the process to the corresponding distribution of the symmetric Wiener process. Section 3 deals with the reflected processes. We consider the first passage of the lower boundary, distribution of the increments of the process and its asymptotic behavior. Finally, in Section 4 we apply the results obtained in the previous sections for studying several characteristics of the queueing system  $M^\kappa|G^\delta|1|B$ , such as busy period, time of the first loss of a customer and the number of customers in the system in transient and stationary regimes.

## 1 Preliminaries and definitions

Let  $\kappa, \delta \in \mathbb{N} = \{1, 2, \dots\}$  be positive independent integer random variables and let  $\eta \in (0, \infty)$  be a positive random variable independent of  $\kappa, \delta$  with the distribution function  $F(x) = \mathbf{P}[\eta \leq x]$ ,  $x \geq 0$ . We will assume that  $\mathbf{E}\kappa$ ,  $\mathbf{E}\delta$ ,  $\mathbf{E}\eta < \infty$ . Introduce the sequences  $\{\eta, \eta'_n\}$ ,  $\{\kappa, \kappa'_n\}$ ,  $\{\delta, \delta'_n\}$ ,  $n \in \mathbb{N}$  of independent identically distributed (inside of each sequence) variables and define the monotone sequences

$$\begin{aligned} \eta_0(x) &= 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta'_1 + \dots + \eta'_n, \quad n \in \mathbb{N}, \\ \kappa_0 &= 0, \quad \kappa_n = \kappa'_1 + \dots + \kappa'_n; \quad \delta_0 = 0, \quad \delta_n = \delta'_1 + \dots + \delta'_n; \quad n \in \mathbb{N}, \end{aligned} \quad (1)$$

where  $\eta_x \in (0, \infty)$  is a random variable with the following distribution function

$$F_x(u) = \mathbf{P}[\eta_x \leq u] = [F(x+u) - F(x)](1 - F(x))^{-1} \quad u \geq 0.$$

Denote by  $\{\pi(t)\}_{t \geq 0} \in \mathbb{Z}^+ = \{0, 1, \dots\}$  a compound Poisson process with the generating function of the form

$$\mathbf{E} \theta^{\pi(t)} = e^{tk(\theta)}, \quad k(\theta) = \mu(\mathbf{E} \theta^\kappa - 1), \quad |\theta| \leq 1,$$

where  $\mu > 0$  is the intensity of the jumps and  $\varkappa$  is a jump size. For all  $t \geq 0$  define a renewal process generated by the random sequence  $\{\eta_n(x)\}_{n \in \mathbb{Z}^+}$  as follows

$$N_x(t) = \max\{n \in \mathbb{Z}^+ : \eta_n(x) \leq t\} \in \mathbb{Z}^+, \quad x \geq 0.$$

Introduce a right-continuous step process for all  $x \geq 0$

$$D_x(t) = \pi(t) - \delta_{N_x(t)} \in \mathbb{Z} = \{0, \pm 1, \dots\}, \quad t \geq 0; \quad D_x(0) = 0. \quad (2)$$

Note, that inter-arrival times of the positive jumps are exponentially distributed with parameter  $\mu$ , the positive jumps themselves are of a random size  $\varkappa$ , and there occur negative jumps of size  $\delta'_n$  at time instants  $\eta_n(x)$ ,  $n \in \mathbb{N}$ . We will call the process  $\{D_x(t)\}_{t \geq 0}$  a difference of the compound Poisson process and a compound renewal process. Observe, that this process is not a Markov process in general. For all  $t \geq 0$  introduce a right-continuous linear component

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \leq t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \geq \eta_x \end{cases} \in \mathbb{R}_+ = [0, \infty), \quad x \geq 0. \quad (3)$$

The process  $\{\eta_x^+(t)\}_{t \geq 0}$  increases linearly on the intervals  $[\eta_n(x), \eta_{n+1}(x))$ ,  $n \in \mathbb{Z}^+$ , it is killed to zero at the points  $\eta_n(x)$ ,  $n \in \mathbb{N}$ , and the value of the process at the instant  $t_0 \geq \eta_x$  is equal to the time elapsed from the moment of the last negative jump of the process (2) till  $t_0$ . We will call the process (3) a linear component (sometimes referred to as the age process). By adding this linear component to the process  $\{D_x(t)\}_{t \geq 0}$  we obtain a right-continuous Markov process

$$\{X_t\}_{t \geq 0} = \{D_x(t), \eta_x^+(t)\}_{t \geq 0} \in \mathbb{Z} \times \mathbb{R}_+, \quad X_0 = \{0, x\}, \quad x \geq 0, \quad (4)$$

which governs the process  $\{D_x(t)\}_{t \geq 0}$ . The process defined in (4) is a Markov process. Note, that it is homogeneous with respect to the first component [18]. This means that if  $X_{t_0} = \{k, u\}$ ,  $k \in \mathbb{Z}$ ,  $u \geq 0$ , then the evolution of the process  $\{X_t\}_{t \geq t_0}$  in the sequel does not depend on the value  $k$  of the first component, and the first positive jump of the process  $\{D_x(t)\}_{t \geq t_0}$  (which is distributed as  $\varkappa$ ) will occur after an exponential period of time with parameter  $\mu$ . The first negative jump of the process  $\{D_x(t)\}_{t \geq t_0}$  (which is distributed as  $\delta$ ) will take place after elapsing of time  $\eta_u$ . This fact will be used constantly when setting up the equations.

Here and in the sequel we assume that the random variable  $\delta \in \mathbb{N}$  is geometrically distributed with parameter  $\lambda \in [0, 1)$ :

$$\mathbf{P}[\delta = n] = (1 - \lambda)\lambda^{n-1}, \quad n \in \mathbb{N}, \quad \mathbf{E}\theta^\delta = \theta \frac{1 - \lambda}{1 - \lambda\theta}, \quad |\theta| \leq 1.$$

This assumption means that the process  $\{D_x(t)\}_{t \geq 0}$  has geometrically distributed negative jumps at time instants  $\{\eta_n(x)\}_{n \in \mathbb{N}}$ . Throughout the article we will use the following notation  $\delta \sim ge(\lambda)$ . In this case it is possible to obtain closed form solutions for the one and the two-sided boundary problems. Our task now is to determine the Laplace transforms of the joint distributions of the upper and lower one-boundary functionals of the process  $\{X_t\}_{t \geq 0}$ . In the sequel we will use the following result.

**Lemma 1.** Let  $\tilde{f}(s) = \mathbf{E}e^{-s\eta}$ . Then for  $s > 0$  the equation

$$\theta - \lambda = (1 - \lambda)\tilde{f}(s - k(\theta)), \quad |\theta| < 1 \quad (5)$$

has a unique solution  $c(s)$  inside the circle  $|\theta| < 1$ . This solution is positive and  $c(s) \in (\lambda, 1)$ . If  $\mathbf{E}[\mathcal{X}], \mathbf{E}[\eta] < \infty$ ,  $\rho = \mu(1 - \lambda)\mathbf{E}[\mathcal{X}]\mathbf{E}[\eta]$ , then for  $\rho > 1$ ,  $\lim_{s \rightarrow 0} c(s) = c \in (\lambda, 1)$ ; and for  $\rho \leq 1$ ,  $\lim_{s \rightarrow 0} c(s) = 1$ .

A detailed proof of an analogous proposition for semi-continuous random walks can be found in the monograph of Spitzer [47]. The reasoning in that proof can be applied to the equation (5) as well (see also Lemma 1 [27]).

Let  $X_0 = \{0, x\}$ ,  $x \in \mathbb{R}_+$ ,  $k \in \mathbb{Z}^+$ . Define

$$\tau_k(x) = \inf\{t : D_x(t) < -k\}, \quad T_k(x) = -D_x(\tau_k(x)) - k, \quad \inf\{\emptyset\} = \infty,$$

i.e. the first overshoot time of the negative level  $-k$  by the process  $\{D_x(t)\}_{t \geq 0}$ . We will use the convention that on the event  $\{\tau_k(x) = \infty\}$   $T_k(x) = \infty$ . Denote  $\mathfrak{B}_k(x) = \{\tau_k(x) < \infty\}$ ,

$$f_k(x, m, s) = \mathbf{E} \left[ e^{-s\tau_k(x)}; T_k(x) = m, \mathfrak{B}_k(x) \right], \quad m \in \mathbb{N}.$$

The Laplace transforms of the joint distribution of the lower one-boundary functionals are determined by means of the following lemma.

**Lemma 2** ([25]). Let  $\{D_x(t)\}_{t \geq 0}$  be the difference of the compound Poisson process and the compound renewal process,  $\delta \sim ge(\lambda)$ . Then

- (i) the Laplace transform of the joint distribution of  $\{\tau_k(x), T_k(x)\}$ ,  $k \in \mathbb{Z}^+$ ,  $x \geq 0$  satisfies the following equality for  $s > 0$ ,  $m \in \mathbb{N}$

$$f_k(x, m, s) = \tilde{f}_x(s - k(c(s))) c(s)^k (1 - \lambda) \lambda^{m-1}, \quad (6)$$

where  $c(s) \in (\lambda, 1)$  is the unique solution of the equation (5) inside the circle  $|\theta| < 1$ ,  $\tilde{f}_x(s) = \mathbf{E} e^{-s\eta_x}$ ,  $\tilde{f}(s) = \mathbf{E} e^{-s\eta} = \tilde{f}_0(s)$ ;

- (ii) if  $\rho > 1$ , then  $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c)) c^k < 1$ , and  $\tau_k(x)$  for all  $k \in \mathbb{Z}^+$ ,  $x \geq 0$  is a defective random variable; if  $\rho \leq 1$ , then  $\mathbf{P}[\tau_k(x) < \infty] = 1$ , and  $\tau_k(x)$  is a proper variable for all  $k \in \mathbb{Z}^+$ ,  $x \geq 0$ .

Observe that for all  $k \in \mathbb{Z}$  the value of the overshoot  $T_k(x)$  does not depend on  $\tau_k(x)$  and it is geometrically distributed  $T_k(x) \sim ge(\lambda)$ .

We now introduce a sequence which will be used to obtain the results in the sequel. The idea to employ this sequence for semi-continuous random walks and semi-continuous Lévy processes is due to Takács [53]. Since the function

$$\tilde{f}_x(s - k(\theta)) = \mathbf{E} \left[ e^{-s\eta_x} \theta^{\pi(\eta_x)} \right] = \sum_{i \in \mathbb{Z}^+} \theta^i \int_0^\infty e^{-st} \mathbf{P}[\eta_x \in dt, \pi(t) = i], \quad |\theta| \leq 1,$$

is analytic inside the unit circle for all  $s, x \geq 0$ , then the function

$$\mathbb{Q}_\theta^s(x) = \frac{(1 - \lambda)\tilde{f}_x(s - k(\theta))}{(1 - \lambda)\tilde{f}(s - k(\theta)) + \lambda - \theta}, \quad s, x \geq 0, \quad |\theta| < c(s) \quad (7)$$

is analytic on the open set  $|\theta| < c(s)$ . In this region it can be represented as a powers series

$$\mathbb{Q}_\theta^s(x) = \sum_{k \in \mathbb{Z}^+} \theta^k Q_k^s(x), \quad s, x \geq 0, \quad |\theta| < c(s).$$

The coefficients of this expansion can be calculated by means of the inversion formula

$$Q_k^s(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{1}{\theta^{k+1}} \frac{(1-\lambda)\tilde{f}_x(s-k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta)) + \lambda - \theta} d\theta, \quad \alpha \in (0, c(s)). \quad (8)$$

We will call the sequence  $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$ ,  $x \geq 0$ , defined by the formula (8) the resolvent sequence of the process  $\{D_x(t)\}_{t \geq 0}$ .

We now explain a probabilistic meaning of this sequence. Introduce a random sequence as follows: (see [26])

$$X_0(x) = 0, \quad X_1(x) = \pi(\eta_x) - \delta, \quad X_{n+1}(x) = X_1(x) + \sum_{i=1}^n X'_i, \quad X_n = X_n(0),$$

where  $X = \pi(\eta) - \delta \in \mathbb{Z}$ ,  $\{X, X'_n\}$ ,  $n \in \mathbb{N}$  is a sequence of i.i.d. random variables. We now define a right-continuous step process in the following way:

$$\{S_x(t)\}_{t \geq 0} = \{X_{N_x(t)}(x)\}_{t \geq 0} \in \mathbb{Z}, \quad S_x(0) = 0, \quad x \in \mathbb{R}_+.$$

The sample paths of the process are constant on the time intervals  $[\eta_n(x), \eta_{n+1}(x))$ ,  $n \in \mathbb{Z}^+$  and there occur jumps at the instants  $\eta_n(x)$ ,  $n \in \mathbb{N}$ . These jumps have the same distribution as  $X \doteq \pi(\eta) - \delta$ , where  $n \in \{2, 3, \dots\}$ , and  $X_1(x) \doteq \pi(\eta_x) - \delta$  for  $n = 1$ . Here and in the sequel we will call the process  $\{S_x(t)\}_{t \geq 0}$  a semi-Markov random walk generated by the sequences  $\{\eta_n(x)\}$ ,  $\{X_n(x)\}$ ,  $n \in \mathbb{Z}^+$ . Let  $S_t^+ = \sup_{u \leq t} S_0(u)$  be the supremum  $\{S_0(t)\}_{t \geq 0}$ . The generating function of  $S_t^+$  was found in [26]:

$$\mathbf{E} \theta^{S_{\nu_s}^+} = \frac{1-\lambda}{1-c(s)} \frac{(1-\tilde{f}(s))(\theta - c(s))}{\theta - \lambda - (1-\lambda)\tilde{f}(s-k(\theta))}, \quad |\theta| \leq 1,$$

where  $\nu_s$  is an exponential variable with parameter  $s > 0$ , independent from the process  $\{S_x(t)\}_{t \geq 0}$ . It follows from (7) and from the latter formula that for  $|\theta| < c(s)$

$$\mathbb{Q}_\theta^s(x) = \frac{1-\tilde{f}(s)}{1-c(s)} \frac{\tilde{f}_x(s-k(\theta))}{c(s)-\theta} \mathbf{E} \theta^{S_{\nu_s}^+}, \quad |\theta| < c(s).$$

Comparing the coefficients of  $\theta^k$ ,  $k \in \mathbb{Z}^+$  in both sides yields

$$Q_k^s(x) = \frac{1-c(s)}{1-\tilde{f}(s)} \sum_{i=0}^k c(s)^{i-k-1} \sum_{j=0}^i \mathbf{E} [e^{-s\eta_x}, \pi(\eta_x) = j] \mathbf{P}[S_{\nu_s}^+ = i-j].$$

Denote by  $\pi^s(\eta_x) \in \mathbb{Z}^+$ ,  $s > 0$  a random variable given by its distribution:

$$\mathbf{P}[\pi^s(\eta_x) = k] = \frac{1}{\tilde{f}_x(s)} \mathbf{E}[e^{-s\eta_x}, \pi(\eta_x) = k], \quad k \in \mathbb{Z}^+.$$

Then the previous equality implies that

$$Q_k^s(x) = \frac{\tilde{f}_x(s)}{1 - \tilde{f}_x(s)} \frac{1 - c(s)}{c(s)^{k+1}} \sum_{i=0}^k c(s)^i \mathbf{P}[\pi^s(\eta_x) + S_{\nu_s}^+ = i], \quad k \in \mathbb{Z}^+.$$

which explains the probabilistic meaning of the resolvent sequence. Asymptotically, one has that  $Q_k^s(x) \sim c(s)^{-k}$  as  $k \rightarrow \infty$ .

Let  $X_0 = \{0, x\}$ ,  $x \geq 0$ ,  $k \in \mathbb{Z}^+$  and introduce upper one-boundary functionals of the process  $\{X_t\}_{t \geq 0}$ :

$$\tau^k(x) = \inf\{t : D_x(t) > k\}, \quad T^k(x) = D_x(\tau^k(x)) - k, \quad \eta^k(x) = \eta_x^+(\tau^k(x))$$

i.e. the instant of the first crossing of the level  $k$  by the process  $\{D_x(t)\}_{t \geq 0}$ , the value of the overshoot across the upper level and the value of the linear component  $\eta_x^+(\cdot)$  at the instant of the first crossing (the time since the last renewal). Denote  $\mathfrak{B}^k(x) = \{\tau^k(x) < \infty\}$ ,

$$f^k(x, dl, m, s) = \mathbf{E}\left[e^{-s\tau^k(x)}; \eta^k(x) \in dl, T^k(x) = m, \mathfrak{B}^k(x)\right], \quad m \in \mathbb{N}.$$

We now determine the upper one-boundary functionals of the process  $\{D_x(t)\}_{t \geq 0}$ . Let  $k \in \mathbb{Z}^+$  and  $\tilde{\tau}^k = \inf\{t : \pi(t) > k\}$ ,  $\tilde{T}^k = \pi(\tilde{\tau}^k) - k$  be the first crossing time through the upper level  $k$  by the compound Poisson process  $\{\pi(t)\}_{t \geq 0}$  and the value of the overshoot at this instant. Denote by

$$\rho_k(t) = \mathbf{P}[\pi(t) = k], \quad \sum_{k=0}^{\infty} \theta^k \rho_k(t) = \mathbf{E} \theta^{\pi(t)} = e^{t\kappa(\theta)}, \quad |\theta| \leq 1,$$

$$p_k^m(dt) = \mathbf{P}[\tilde{\tau}^k \in dt, \tilde{T}^k = m] = \mu \sum_{i=0}^k \rho_i(t) \mathbf{P}[\varkappa = k - i + m] dt, \quad m \in \mathbb{N}.$$

**Lemma 3** ([25]). *Let  $\{D_x(t)\}_{t \geq 0}$  be the difference of the compound Poisson process and the compound renewal process,  $\delta \sim ge(\lambda)$ ,  $x \geq 0$ ,  $k \in \mathbb{Z}^+$  and  $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$ , be the resolvent sequence of the process  $\{D_x(t)\}_{t \geq 0}$ , given by (8).*

(i) *the Laplace transforms of the joint distribution of  $\{\tau^k(x), \eta^k(x), T^k(x)\}$  satisfy the following equality*

$$\begin{aligned} f^k(x, dl, m, s) &= e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x)) \\ &+ \Phi_\lambda^s(0, dl, m) Q_k^s(x) - e^{-sl} [1 - F(l)] \sum_{i=0}^k Q_i^s(x) p_{k-i}^m(dl), \end{aligned} \quad (9)$$

where  $\Phi_\lambda^s(0, dl, m) = e^{-sl} [1 - F(l)] \sum_{k=0}^{\infty} c(s)^k p_k^m(dl)$ ;

- (ii) the Laplace transform of the first crossing time through the upper level  $k$  by the process  $\{D_x(t)\}_{t \geq 0}$  are such that for all  $k \in \mathbb{Z}^+$ ,  $s, x \geq 0$

$$f^k(x, s) = \mathbf{E}e^{-s\tau^k(x)} = 1 - \frac{s}{s - k(c(s))} \frac{Q_k^s(x)}{1 - \lambda} - A_x^k(s), \quad (10)$$

where  $A_x^k(s) = \sum_{i=0}^k \tilde{\rho}_i(s) [1 - Q_{k-i}^s(x)(1 - \lambda)^{-1}]$ ,  $\tilde{\rho}_k(s) = s \int_0^\infty e^{-st} \rho_k(t) dt$ ;

- (iii) for  $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$  and  $\rho < 1$ ,  $\tau^k(x)$  is a defective random variable and

$$\mathbf{P}[\tau^k(x) < \infty] = 1 - (1 - \rho)(1 - \lambda)^{-1} Q_k(x) < 1, \quad k \in \mathbb{Z}^+ \quad x \geq 0,$$

where  $\{Q_k(x)\}_{k \in \mathbb{Z}^+}$ ,  $x \geq 0$  is the resolvent sequence of the process  $\{D_x(t)\}_{t \geq 0}$ , given by (8) for  $s = 0$ :

$$Q_k(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{d\theta}{\theta^{k+1}} \frac{(1 - \lambda)\tilde{f}_x(-k(\theta))}{(1 - \lambda)\tilde{f}(-k(\theta)) + \lambda - \theta}, \quad \alpha \in (0, c(0)); \quad (11)$$

if  $\rho \geq 1$ , then for all  $k \in \mathbb{Z}^+$ ,  $x \geq 0$   $\tau^k(x)$  is a proper random variable.

Along with expression (11) there exists another way to calculate  $Q_k(x)$ , which is more applicable from practical point of view. We will now derive the recurrent formula for  $Q_k(x)$ . It follows from (7) for  $s, \theta = 0$  that

$$Q_0(x) = (1 - \lambda)(\lambda + (1 - \lambda)f_0)^{-1} f_0(x),$$

where for all  $k \in \mathbb{Z}^+$

$$f_k(x) = \mathbf{P}[\pi(\eta_x) = k] = \int_0^\infty \mathbf{P}[\eta_x \in dt, \pi(t) = k], \quad f_k = f_k(0).$$

Again, it follows from (7) for  $s = 0$  that

$$(1 - \lambda)\tilde{f}_x(-k(\theta)) = (1 - \lambda)\tilde{f}(-k(\theta))Q_\theta(x) + (\lambda - \theta)Q_\theta(x).$$

Comparing the coefficients of  $\theta^k$ ,  $k \in \mathbb{N}$  in both sides implies that

$$(1 - \lambda)f_k(x) = (1 - \lambda) \sum_{i=0}^k Q_i(x)f_{k-i} + \lambda Q_k(x) - Q_{k-1}(x).$$

Combining like terms yields

$$(\lambda + (1 - \lambda)f_0) Q_k(x) = (1 - \lambda)f_k(x) + Q_{k-1}(x) - (1 - \lambda) \sum_{i=0}^{k-1} Q_i(x)f_{k-i}.$$

The latter formula is a recurrent relation which allows to calculate successively the terms  $Q_k(x)$  given the previous terms  $Q_0(x), \dots, Q_{k-1}(x)$ . For instance, given the expression for  $Q_0(x)$  one finds that

$$Q_1(x) = \frac{1 - \lambda}{\lambda + (1 - \lambda)f_0} \left[ f_1(x) + \frac{1 - (1 - \lambda)f_0}{\lambda + (1 - \lambda)f_0} f_0(x) \right].$$

The knowledge of the one-boundary characteristics of the process allows us to solve the two-sided problems, which is the aim of the following section.



## 2 Two-sided problems for the process $\{D_x(t)\}_{t \geq 0}$

Let  $B \in \mathbb{Z}^+$  be fixed,  $k \in \overline{0, B}$ ,  $r = B - k$ ,  $X_0 = \{0, x\}$ ,  $x \geq 0$ , and introduce the random variable

$$\chi_r^B(x) = \inf\{t : D_x(t) \notin [-r, k]\} \stackrel{\text{def}}{=} \chi$$

the first exit time from the interval  $[-r, k]$  by the process  $\{D_x(t)\}_{t \geq 0}$ . This random variable takes values from a countable set  $\{\xi_n, n \in \mathbb{N}\} \cup \{\eta_n(x), n \in \mathbb{N}\}$ , and it is a Markov time of the process  $\{X_t\}_{t \geq 0}$  ( $\xi_k$  are the instants of jumps of the process  $\pi(t)$ .) Note, that the exit from the interval can occur either through the upper boundary  $k$ , or through the lower boundary  $-r$ . In view of this remark introduce the events

$\mathfrak{A}^k = \{D_x(\chi) > k\}$ , i.e. the process  $\{D_x(t)\}_{t \geq 0}$  exits the interval  $[-r, k]$  through the upper boundary  $k$ ;

$\mathfrak{A}_r = \{D_x(\chi) < -r\}$ , i.e. the process  $\{D_x(t)\}_{t \geq 0}$  exits the interval  $[-r, k]$  through the lower boundary  $-r$ . Denote by

$$T = (D_x(\chi) - k)\mathbf{I}_{\mathfrak{A}^k} + (-D_x(\chi) - r)\mathbf{I}_{\mathfrak{A}_r}, \quad L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0 \cdot \mathbf{I}_{\mathfrak{A}_r}, \quad \mathbf{P}[\mathfrak{A}^k + \mathfrak{A}_r] = 1$$

the value of the overshoot through the boundaries of the interval  $[-r, k]$  by the process  $\{D_x(t)\}_{t \geq 0}$  and the value of the linear component at the instant of the first exit (the time since the last renewal), where  $\mathbf{I}_{\mathfrak{A}} = \mathbf{I}_{\mathfrak{A}}(\omega)$  is the indicator function of the event  $\mathfrak{A}$ . Denote

$$V^k(x, dl, m, s) = \mathbf{E} \left[ e^{-s\chi}; L \in dl, T = m, \mathfrak{A}^k \right], \quad V_r(x, m, s) = \mathbf{E} \left[ e^{-s\chi}; T = m, \mathfrak{A}_r \right].$$

**Theorem 1** ([25]). *Let  $\{D_x(t)\}_{t \geq 0}$  be the difference of the compound Poisson process and the renewal process (2),  $\delta \sim ge(\lambda)$ ,  $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$ , be the resolvent sequence of the process given by (8),  $Q_k^s \stackrel{\text{def}}{=} Q_k^s(0)$ . Then*

- (i) *the Laplace transforms of the joint distribution of  $\{\chi, L, T\}$  satisfy the following equalities for all  $x, s \geq 0$ ,  $m \in \mathbb{N}$*

$$\begin{aligned} V_r(x, m, s) &= \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} (1 - \lambda) \lambda^{m-1}, \\ V^k(x, dl, m, s) &= f^k(x, dl, m, s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \mathbf{E} f^{\delta+B}(0, dl, m, s), \end{aligned} \quad (12)$$

where the function  $f^k(x, dl, m, s)$  is given by (9),

$$\begin{aligned} \mathbf{E} Q_{\delta+B}^s &= \sum_{k \in \mathbb{N}} (1 - \lambda) \lambda^{k-1} Q_{k+B}^s, \\ \mathbf{E} f^{\delta+B}(0, dl, m, s) &= \sum_{k \in \mathbb{N}} (1 - \lambda) \lambda^{k-1} f^{k+B}(0, dl, m, s); \end{aligned}$$

(ii) for the Laplace transforms of the first exit time  $\chi$  from the interval by the process  $\{D_x(t)\}_{t \geq 0}$  the following formulae hold

$$\mathbf{E} [e^{-s\chi}; \mathfrak{A}_r] = \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s}, \quad (13)$$

$$\mathbf{E} [e^{-s\chi}; \mathfrak{A}^k] = 1 - A_x^k(s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \left(1 - \mathbf{E} A_0^{\delta+B}(s)\right),$$

where  $\mathbf{E} A_0^{\delta+B}(s) = \sum_{k \in \mathbb{N}} (1 - \lambda) \lambda^{k-1} A_0^{k+B}(s)$ ;

(iii) the probabilities of the exit from the interval through the upper and the lower boundary by the process  $\{D_x(t)\}_{t \geq 0}$  are given by

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{\mathbf{E} Q_{\delta+B}}, \quad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{\mathbf{E} Q_{\delta+B}},$$

where the resolvent sequence of the process  $\{Q_k(x)\}_{k \in \mathbb{Z}^+}$ ,  $x \geq 0$  is defined by (11),  $Q_k \stackrel{\text{def}}{=} Q_k(0)$ .

Denote by  $\nu_s \sim \exp(s)$  an exponential random variable with parameter  $s > 0$  independent of the process  $D_x(t)$ . For  $k \in \mathbb{Z}^+$ ,  $x \geq 0$  define  $D_x^+(t) = \sup_{[0,t]} D_x(\cdot)$ ,  $D_x^-(t) = \inf_{[0,t]} D_x(\cdot)$  the running maximum and minimum of the process. Our aim is to determine the joint distribution of  $\{D_x^-(t), D_x(t), D_x^+(t)\}$ . In order to do this, we will require the joint distribution of  $\{D_x(\nu_s), D_x^+(\nu_s)\}$ .

**Lemma 4.** Let  $k \in \mathbb{Z}^+$  and  $E_k^+(x, z, s) = \mathbf{E} [z^{D_x(\nu_s)}; D_x^+(\nu_s) \leq k]$ ,  $|z| \geq 1$  be the generating function of the joint distribution of  $\{D_x(\nu_s), D_x^+(\nu_s)\}$ . Then

(i) the generating function  $E_k^+(x, z, s)$  is such that

$$E_k^+(x, z, s) = z^k A_x^k(s) + (1 - z) \sum_{i=0}^{k-1} z^i A_x^i(s) + z^k Q_k^s(x) \mathbb{E}_{\lambda/z}^s(0, z), \quad (14)$$

where

$$\mathbb{E}_{\lambda/z}^s(0, z) = \frac{s(1 - \lambda)^{-1}}{s - k(c(s))} \frac{1 - c(s)}{1 - c(s)/z};$$

(ii) the joint distribution  $\mathfrak{E}_k^+(x, u, s) = \mathbf{P} [D_x(\nu_s) \leq u, D_x^+(\nu_s) \leq k]$ ,  $u \in \overline{-\infty, k}$  satisfies the following equality

$$\mathfrak{E}_k^+(x, u, s) = A_x^u(s) + \frac{s(1 - \lambda)^{-1}}{s - k(c(s))} c(s)^{k-u} Q_k^s(x), \quad A_x^u(s) = 0, \quad u < 0; \quad (15)$$

(iii) under the condition (A)

$$(A) \quad \rho = (1 - \lambda) \mu \mathbf{E} \eta \mathbf{E} \varkappa = 1, \quad \sigma^2 = \mu \left[ \mathbf{E} \varkappa (\varkappa - 1) + \frac{\mathbf{E} \varkappa \mathbf{E} \eta^2}{(1 - \lambda)(\mathbf{E} \eta)^2} \right] < \infty,$$

the following limiting equality holds as  $B \rightarrow \infty$ ,  $k > 0$

$$\mathbf{P}[D_x(tB^2) \leq [uB], D_x^+(tB^2) \leq [kB]] \rightarrow \frac{1}{\sigma\sqrt{2\pi t}} \int_{-u}^{2k-u} e^{-v^2/2\sigma^2 t} dv,$$

where  $[a]$  is the integer part of the number  $a$ ,  $u \leq k$ .

*Proof.* In view of the total probability law, homogeneity of the process  $X_t$  with respect to the first component, Markov property of  $\eta_1(x)$  we can write for the function  $E_k^+(x, z, s)$ ,  $k \in \mathbb{Z}^+$ ,  $x \geq 0$  the following equation

$$\begin{aligned} E_k^+(x, z, s) &= s \int_0^\infty e^{-st} \mathbf{P}[\eta_x > t] \mathbf{E}[z^{\pi(t)}; \pi(t) \leq k] dt + \\ &+ \int_0^\infty e^{-su} \sum_{v=0}^k \mathbf{P}[\eta_x \in du, \pi(u) = v] z^v \sum_{r=1}^\infty (1-\lambda) \lambda^{r-1} z^{-r} E_{k-v+r}^+(0, z, s). \end{aligned} \quad (16)$$

Introduce the generating function  $\mathbb{E}_\theta^s(x, z) = \sum_{k \in \mathbb{Z}^+} \theta^k E_k^+(x, z, s)$ ,  $|\theta| < 1$ . Multiplying (16) by  $\theta^k$  and summing over  $k \in \mathbb{Z}^+$ , we derive the following equation for the function  $\mathbb{E}_\theta^s(x, z)$

$$\begin{aligned} \mathbb{E}_\theta^s(x, z) &= \frac{s}{1-\theta} \frac{1 - \tilde{f}_x(s - k(z\theta))}{s - k(z\theta)} + \\ &+ \tilde{f}_x(s - k(z\theta)) \frac{(1-\lambda)}{\lambda - z\theta} \left[ \mathbb{E}_{\lambda/z}^s(0, z) - \mathbb{E}_\theta^s(0, z) \right], \quad |\theta| < 1, |z| \geq 1. \end{aligned} \quad (17)$$

Letting  $x = 0$  in the latter equation yields

$$\begin{aligned} \mathbb{E}_\theta^s(0, z) &= \frac{z\theta - \lambda}{(1-\lambda)\tilde{f}(s - k(z\theta)) + \lambda - z\theta} \times \\ &\times \left[ \tilde{f}(s - k(z\theta)) \frac{1-\lambda}{z\theta - \lambda} \mathbb{E}_{\lambda/z}^s(0, z) - \frac{s}{1-\theta} \frac{1 - \tilde{f}(s - k(z\theta))}{s - k(z\theta)} \right]. \end{aligned}$$

The function which enters the left-hand side of this equation is analytic in  $|\theta| < 1$ . In view of Lemma 1 it has denominator of the right-hand side has a simple zero in  $\theta = c(s)/z$ . Hence, the nominator of right-hand side should also have the simple zero. Letting  $\theta = c(s)/z$  in the nominator we find the function  $\mathbb{E}_{\lambda/z}^s(0, z)$

$$\mathbb{E}_{\lambda/z}^s(0, z) = \frac{s(1-\lambda)^{-1}}{s - k(c(s))} \frac{1 - c(s)}{1 - c(s)/z}, \quad |z| \geq 1.$$

Employing the definition of the resolvent (7) and substituting the expression for  $\mathbb{E}_\theta^s(0, z)$  into (17), we get

$$\mathbb{E}_\theta^s(x, z) = \mathbb{A}_x^{z\theta}(s) + \theta \frac{1-z}{1-\theta} \mathbb{A}_x^{z\theta}(s) + \mathbb{Q}_{z\theta}^s(x) \mathbb{E}_{\lambda/z}^s(0, z), \quad (18)$$

where

$$\mathbb{A}_x^{z\theta}(s) = \sum_{k=0}^\infty (z\theta)^k A_x^k(s) = \frac{s}{s - k(z\theta)} \left( \frac{1}{1 - z\theta} - \frac{1}{1 - \lambda} \mathbb{Q}_{z\theta}^s(x) \right).$$

Using the definition of the resolvent (8) and comparing the coefficients of  $\theta^k$ ,  $k \in \mathbb{Z}^+$  in both sides of (18) implies that

$$E_k^+(x, z, s) = z^k A_x^k(s) + (1 - z) \sum_{i=0}^{k-1} z^i A_x^i(s) + z^k Q_k^s(x) \mathbb{E}_{\lambda/z}^s(0, z),$$

i.e. the equality (14) of the lemma. Comparing the coefficients of  $z^i$ ,  $i \in \overline{-\infty, k}$  in both sides of the latter equality, we find

$$\begin{aligned} \mathbf{P}[D_x(\nu_s) = i, D_x^+(\nu_s) \leq k] &= \\ &= A_x^i(s) - A_x^{i-1}(s) + \frac{s}{s - k(c(s))} \frac{1 - c(s)}{1 - \lambda} c(s)^{k-i} Q_k^s(x), \quad i \leq k, \end{aligned}$$

where  $A_x^i(s) = 0$ , for  $i < 0$ . The latter formula implies (15). Denote  $\tilde{e}_k^t(x, u, B) = \mathbf{P}[D_x(tB^2) \leq [uB], D_x^+(tB^2) \leq [kB]]$ . It is clear that

$$\lim_{B \rightarrow \infty} \int_0^\infty e^{-st} \tilde{e}_k^t(x, u, B) dt = \frac{1}{s} \lim_{B \rightarrow \infty} \mathfrak{E}_{[kB]}^{s/B^2}(x, [uB]).$$

To proceed further, we need the following limiting equalities (see [26])

$$\begin{aligned} c(s/B^2) &= 1 - B^{-1} \sqrt{2s}/\sigma + o(B^{-1}), \\ \lim_{B \rightarrow \infty} B^{-1} Q_{[kB]}^{s/B^2}(x) &= \frac{2 \operatorname{sh}(k \sqrt{2s}/\sigma)}{\sigma \sqrt{2s} \mathbf{E} \eta} = \lim_{B \rightarrow \infty} B^{-1} \mathbf{E} Q_{\delta+[kB]}^{s/B^2}, \\ \lim_{B \rightarrow \infty} A_x^{[kB]}(s/B^2) &= 1 - \operatorname{ch}(k \sqrt{2s}/\sigma) = \lim_{B \rightarrow \infty} A_0^{\delta+[kB]}(s/B^2). \end{aligned} \quad (19)$$

In view of these equalities and of the formula (15) we derive

$$\begin{aligned} \lim_{B \rightarrow \infty} \int_0^\infty e^{-st} \tilde{e}_k^t(x, u, B) dt &= s^{-1} \mathbf{I}_{\{u < 0\}} \left( e^{u \sqrt{2s}/\sigma} / 2 - e^{-(2k-u) \sqrt{2s}/\sigma} / 2 \right) \\ &+ s^{-1} \mathbf{I}_{\{u \in [0, k]\}} \left( 1 - e^{-u \sqrt{2s}/\sigma} / 2 - e^{-(2k-u) \sqrt{2s}/\sigma} / 2 \right), \quad u \leq k. \end{aligned}$$

Denote by  $w_{\{t \geq 0\}}$  the symmetric Wiener process with the dispersion  $\sigma$  and by  $\tau^a = \inf\{t : w_t \geq a\}$  the first passage time of the level  $a \in \mathbb{R}_+$ . The Lévy formula  $\mathbf{P}[\tau \leq t] = 2\mathbf{P}[w_t \geq a]$  implies for the Laplace transforms that

$$\frac{1}{s} e^{-a \sqrt{2s}/\sigma} = 2 \int_0^\infty e^{-st} \mathbf{P}[w_t \geq a] dt.$$

Employing the latter formula to invert the Laplace transforms in the previous equality, we derive the second limiting formula of the theorem.  $\square$

Let  $k, r \in \mathbb{Z}^+$ ,  $u \in \overline{-r, k}$  and denote by

$$\begin{aligned} \tilde{e}_{r,k}^t(x, u) &= \mathbf{P}[-r \leq D_x^-(t), D_x^-(t) \leq u, D_x^+(t) \leq k] = \mathbf{P}[D_x(t) \leq u, \chi_x^B(r) > t], \\ \mathfrak{E}_{r,k}^s(x, u) &= \tilde{e}_{r,k}^{\nu_s}(x, u) = s \int_0^\infty e^{-st} \tilde{e}_{r,k}^t(x, u) dt \end{aligned}$$

the joint distribution of  $\{D_x^-(t), D_x(t), D_x^+(t)\}$  and its Laplace transform.

**Theorem 2.** Let  $\nu_s \sim \exp(s)$  be an exponential random variable independent of the process  $D_x(t)$ ,  $B = r + k$ . Then

(i) the joint distribution of  $\{D_x^-(\nu_s), D_x(\nu_s), D_x^+(\nu_s)\}$  is such that

$$\mathfrak{E}_{r,k}^s(x, u) = A_x^u(s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \mathbf{E} A_x^{\delta+r+u}(s), \quad u \in \overline{-r, k}, \quad (20)$$

where  $\mathbf{E} A_x^{\delta+r+u}(s) = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} A_x^{i+r+u}(s)$ ;

(ii) under the condition (A) and  $r \in (0, 1)$ ,  $k = 1 - r$  the joint distribution  $\tilde{e}_{[rB], [kB]}^{tB^2}(x, [uB])$  weakly converges as  $B \rightarrow \infty$  to the joint distribution

$$\mathbf{P} \left[ -r \leq \inf_{v \leq t} w_v, w_t \leq u, \sup_{v \leq t} w_v \leq k \right], \quad u \in [-r, k]$$

of the infimum, the supremum and the value of the symmetric Wiener process with the dispersion  $\sigma$ . In addition, the following limiting equality holds

$$\lim_{B \rightarrow \infty} \tilde{e}_{[rB], [kB]}^{tB^2}(x, [uB]) = \frac{4}{\pi} \sum_{n \in \mathbb{N}} \frac{e^{-\frac{t}{2}(\pi n \sigma)^2}}{n} \sin(r\pi n) \sin^2 \left( \frac{r+u}{2} n\pi \right). \quad (21)$$

*Proof.* The total probability law, homogeneity of the process  $X_t$  with respect to the first component, Markov property of  $\chi_r^B(x)$  for all  $k, r \in \mathbb{Z}^+$ ,  $x \geq 0$  imply the following equation for  $|z| \geq 1$

$$\begin{aligned} \mathbf{E} \left[ z^{D_x(\nu_s)}; D_x^+(\nu_s) \leq k \right] &= \mathbf{E} \left[ z^{D_x(\nu_s)}; \chi_r^B(x) > \nu_s \right] + \\ &+ \sum_{i=1}^{\infty} \mathbf{E} \left[ e^{-s\chi_r^B(x)}; T = i, \mathfrak{A}_r \right] z^{-(r+i)} \mathbf{E} \left[ z^{D_x(\nu_s)}; D_x^+(\nu_s) \leq i + B \right], \end{aligned} \quad (22)$$

where  $B = k + r$ . This equation for the case of a spectrally one-sided Lévy process was derived in [22], and for the general Lévy process in [29]. Let us briefly explain the equation (22). The increments of the process  $D_x(t)$  on the interval  $[0, \nu_s]$  without the intersection of the level  $k$  (the left-hand side) can be realized either on the sample paths of the process which do not cross the negative level  $-r$  (the first term of the right-hand side) or on the sample paths which do cross the level  $-r$  and then the further evolution of the process is nothing but its probabilistic copy on  $[0, \nu_s]$  (the second term). In view of (22) and (12), (14) we find for the function  $E_{r,k}^s(x, z) = \mathbf{E} [z^{D_x(\nu_s)}; \chi_r^B(x) > \nu_s]$  that

$$\begin{aligned} E_{r,k}^s(x, z) &= E_k^+(x, z, s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} (1 - \lambda) \sum_{i \in \mathbb{N}} \lambda^{i-1} z^{-(r+i)} E_{i+B}^+(0, z, s) = \\ &= z^k A_x^k(s) + (1 - z) \sum_{i=0}^{k-1} z^i A_x^i(s) + \\ &+ z^k \frac{Q_k^s(x)}{\tilde{Q}_B^s(\lambda)} \frac{(1 - \lambda) \tilde{A}_0^B(s, \lambda) - (1 - z)(\lambda/z)^{B+1} \tilde{A}_0^B(s, z)}{1 - \lambda/z}, \end{aligned} \quad (23)$$

where  $\check{A}_0^B(s, z) = \sum_{i=0}^B z^i A_0^i(s)$ ,  $\hat{Q}_B^s(\lambda) = \sum_{i=B+1}^{\infty} \lambda^i Q_i^s(0)$ . The formula (13) yields

$$\mathbf{P} [\chi_r^B(x) > \nu_s] = A_x^k(s) + \frac{Q_k^s(x)}{\hat{Q}_B^s(\lambda)} \check{A}_x^B(s, \lambda).$$

It is not difficult to derive the following equality

$$\sum_{u=-r}^k z^u \mathfrak{E}_{r,k}^s(x, u) = \frac{1}{1-z} \left( E_{r,k}^s(x, z) - z^{k+1} \mathbf{P} [\chi_r^B(x) > \nu_s] \right).$$

The right-hand side of (23) implies that

$$\sum_{u=-r}^k z^u \mathfrak{E}_{r,k}^s(x, u) = \sum_{u=0}^k z^u A_x^u(s) + z^k \frac{Q_k^s(x)}{\hat{Q}_B^s(\lambda)} \sum_{i=0}^B (\lambda/z)^i \sum_{j=0}^{B-i} \lambda^j A_0^j(s).$$

Comparing the coefficients of  $z^u$ ,  $u \in \overline{-r, k}$ , we find

$$\mathfrak{E}_{r,k}^s(x, u) = A_x^u(s) + \frac{Q_k^s(x)}{\hat{Q}_B^s(\lambda)} \lambda^{k-u} \sum_{i=0}^{r+u} \lambda^i A_0^i(s).$$

Since

$$\mathbf{E} Q_{\delta+B}^s = (1-\lambda) \lambda^{-B-1} \hat{Q}_B^s(\lambda), \quad \sum_{i=0}^{\infty} \lambda^i A_0^i(s) = 0,$$

one can see that the previous equality is the formula (20). Let us verify (21). It is clear that

$$s \int_0^{\infty} e^{-st} \tilde{e}_{[rB], [kB]}^{tB^2}(x, [uB]) dt = \mathfrak{E}_{[rB], [kB]}^{s/B^2}(x, [uB]), \quad k \in (0, 1) \quad r = 1 - k,$$

where the function  $\mathfrak{E}_{r,k}^s(x, u)$  is determined by (20). Thus,

$$\begin{aligned} \lim_{B \rightarrow \infty} \int_0^{\infty} e^{-st} \tilde{e}_{[rB], [kB]}^{tB^2}(x, [uB]) dt &= \frac{1}{s} \lim_{B \rightarrow \infty} \mathfrak{E}_{[rB], [kB]}^{s/B^2}(x, [uB]) \stackrel{\text{def}}{=} e^*(s) = \\ &= \frac{1}{s} \left[ 1 - \text{ch} \left( \frac{u^+}{\sigma} \sqrt{2s} \right) \right] + \frac{1}{s} \frac{\text{sh} k \sqrt{2s}/\sigma}{\text{sh} \sqrt{2s}/\sigma} \left[ \text{ch} \left( \frac{r+u}{\sigma} \sqrt{2s} \right) - 1 \right], \end{aligned} \quad (24)$$

where  $u^+ = \max(0, u)$ . In order to compute this limit we used the formulae (19). Note, that the inversion of the Laplace transform in the right-hand side of (24) this equality was found in [22] and resulted into the following formula ( $\alpha > 0$ )

$$\begin{aligned} \mathbf{P} \left[ -r \leq \inf_{v \leq t} w_v, w_t \leq u, \sup_{v \leq t} w_v \leq k \right] &= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} e^*(s) ds = \\ &= \frac{4}{\pi} \sum_{n \in \mathbb{N}} \frac{e^{-\frac{t}{2}(\pi n \sigma)^2}}{n} \sin(r\pi n) \sin^2 \left( \frac{r+u}{2} n\pi \right), \quad u \in [-r, k]. \end{aligned}$$

Therefore, we established the weak convergence of the joint distribution  $\tilde{e}_x^t(u, B)$  as  $B \rightarrow \infty$  to the corresponding distribution of the Wiener process and also verified the formula (21).  $\square$

### 3 Reflections from the boundary

Denote by  $D_x^r(t) = r + D_x(t)$ ,  $t \geq 0$  the process starting from  $r \in \mathbb{Z}$  when  $\eta_x^+(0) = x \geq 0$ . Let  $B \in \mathbb{Z}^+$  and for all  $t \geq 0$  we define right-continuous processes reflected at the boundary B as follows

$$\overline{D}_r^B(x, t) = D_x^r(t) - \max \left\{ 0, \sup_{[0, t]} D_x^r(\cdot) - B \right\} \in \overline{-\infty, B}, \quad r \in \overline{-\infty, B}. \quad (25)$$

The first reflection from the upper boundary  $B$  of the process  $\overline{D}_r^B(x, t)$  takes place at  $\tau^{B-r}(x)$ . Then the process stays at the boundary for some random time  $\eta_l$ , where  $l = \eta_x^+(\tau^{B-r}(x))$ . At the instant  $t = \tau^{B-r}(x) + \eta_l$  the process is reflected to a random state  $B - \delta$ . In the sequel the evolution of the process  $\overline{D}_r^B(x, t)$  is a probabilistic copy of its evolution on  $[0, \tau^{B-r}(x) + \eta_l)$ . It is worth noticing that reflections from the boundaries reflected by infimum (supremum) were introduced by Lévy for a standard Wiener process. Applying the symmetry principle and the mirror reflection principle Lévy determined the distributions of the boundary functionals of the reflected standard Wiener process. We will show that these distributions are the weak limit distributions for the reflected process after an appropriate scaling of time and space.

#### 3.1 Passage of the lower boundary

We now define the boundary functionals for process (25). For  $r \in \overline{0, B}$  denote

$$\overline{\tau}_r^B(x) = \inf \{t : \overline{D}_r^B(x, t) < 0\} \stackrel{\text{def}}{=} \overline{\tau}, \quad \overline{T}_r^B(x) = -\overline{D}_r^B(\overline{\tau}) \stackrel{\text{def}}{=} \overline{T}, \quad r \in [0, B]$$

the first crossing time of the lower level 0 by the process  $\overline{D}_r^B(x, t)$  and the value of the overshoot at this instant. Note, that these boundary functionals were studied in [40] for the reflected Lévy processes generated by infimum (supremum). It is worth noticing that in this article the asymptotic expansions for the distributions of the characteristics of the process were determined for the reflected Lévy processes obeying the two-boundary Cramer's conditions.

The reflected spectrally one-sided Lévy processes generated by the infimum (supremum) of the process were considered in [3], [41]. An interesting application in queueing theory for the spectrally one-sided Lévy process reflected by its infimum was given in [6].

**Theorem 3.** *Let  $\{\overline{D}_r^B(x, t)\}_{t \geq 0}$  be the reflected processes defined by (25),  $B \in \mathbb{Z}^+$ ,  $r \in \overline{0, B}$ ,*

$$V^k(x, dl, m, s) = \mathbf{E} \left[ e^{-sx}; L \in dl, T = m, \mathfrak{A}^k \right], \quad V_r(x, m, s) = \mathbf{E} \left[ e^{-sx}; T = m, \mathfrak{A}_r \right]$$

*the Laplace transforms of the joint distribution of  $\{\chi_r^B(x), L, T\}$  of the process  $\{D_x(t)\}_{t \geq 0}$  [25]. Then*

- (i) if  $\delta \in \mathbb{N}$  (an arbitrarily distributed non-negative variable), then the Laplace transform of the joint distribution of  $\{\bar{\tau}, \bar{T}\}$  is such that for  $(m \in \mathbb{N})$

$$\begin{aligned} \bar{v}_r^s(x, m) &= \mathbf{E} \left[ e^{-s\bar{\tau}_r^B(x)}; \bar{T} = m \right] = V_r(x, m, s) \\ &+ \frac{a^{B-r}(x)}{1 - A(0)} \left[ \mathbf{P}[\delta = m + B] + \sum_{i=1}^B \mathbf{P}[\delta = i] V_{B-i}(0, m, s) \right], \end{aligned} \quad (26)$$

where  $V^k(x, dl, s) = \sum_{m=1}^{\infty} V^k(x, dl, m, s)$ ,

$$a^k(x) = \int_0^{\infty} V^k(x, dl, s) \tilde{f}_l(s), \quad A(x) = \sum_{k=1}^B \mathbf{P}[\delta = k] a^k(x);$$

- (ii) if  $\delta \sim ge(\lambda)$ , then the following equalities hold

$$\bar{v}_r^s(x, m) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s)) S_{B-r-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s)) \mathbf{E} S_{\delta+B-1}^s} (1 - \lambda) \lambda^{m-1}, \quad r \in \overline{0, B}, \quad (27)$$

where  $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$ ,  $\mathbf{E} S_{\delta+B-1}^s = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} S_{i-1+B}^s(0)$ ; the random variable  $\bar{\tau}_r^B(x)$  is proper ( $\mathbf{P}[\bar{\tau}_r^B(x) < \infty] = 1$ ) and

$$\mathbf{E} \bar{\tau}_r^B(x) = \mathbf{E} \eta_x - \mathbf{E} \eta + \mathbf{E} \eta [\mathbf{E} S_{\delta+B-1} - S_{B-r-1}(x)] < \infty, \quad (28)$$

where  $S_k(x) = S_k^0(x)$ ,  $\mathbf{E} S_{\delta+B} = \mathbf{E} S_{\delta+B}^0$ ;

- (iii) under the conditions (A) the following equality is valid

$$\lim_{B \rightarrow \infty} \mathbf{E} e^{-s\bar{\tau}_{[rB]}^B(x)/B^2} = \frac{\text{ch}(k\sqrt{2s}/\sigma)}{\text{ch}(\sqrt{2s}/\sigma)}, \quad r \in (0, 1), \quad k = 1 - r.$$

*Proof.* Let us verify the formula (26). It follows from the definition of the process  $\bar{D}_r^B(x, t)$  (25), the total probability law and the Markov property of  $\chi$ ,  $\eta_n(x)$  that the following system of the linear integral equations holds

$$\begin{aligned} \bar{v}_r^s(x, m) &= V_r(x, m, s) + \int_0^{\infty} V^k(x, dl, s) \bar{v}_B^s(l, m), \quad k = B - r, \\ \bar{v}_B^s(x, m) &= \tilde{f}_x(s) \mathbf{P}[\delta = m + B] + \tilde{f}_x(s) \sum_{r=1}^B \mathbf{P}[\delta = r] \bar{v}_{B-r}^s(0, m). \end{aligned}$$

This system is similar to a system of linear equations with two unknowns and can be solved analogously. Substituting the expression for  $\bar{v}_B^s(x, m)$  from the second equation into the first one, we find that

$$\bar{v}_r^s(x, m) = V_r(x, m, s) + a^k(x) \mathbf{P}[\delta = m + B] + a^k(x) \sum_{r=1}^B \mathbf{P}[\delta = r] \bar{v}_{B-r}^s(0, m).$$



Letting  $x = 0$  in the latter equation after calculations yields

$$\begin{aligned} \sum_{r=1}^B \mathbf{P}[\delta = r] \bar{v}_{B-r}^s(0, m) &= -\mathbf{P}[\delta = m + B] \\ &+ \left[ \mathbf{P}[\delta = m + B] + \sum_{k=1}^B \mathbf{P}[\delta = k] V_{B-k}(0, du, s) \right] (1 - A(0))^{-1}. \end{aligned}$$

Inserting the right-hand side of this quality in the previous one, we get (26). In case when  $\delta \sim ge(\lambda)$  the formula (26) takes a more simple form. The first formula of (12) and (26) imply that  $\bar{T}_r^B(x) \sim ge(\lambda)$  for any  $r \in \overline{0, B}$ . Summing over  $m \in \mathbb{N}$  both sides of (26), we find for the function  $\bar{v}_r^s(x) = \mathbf{E}e^{-s\bar{\tau}_r^B(x)}$  that

$$\bar{v}_r^s(x) = V_r(x, s) + \frac{a^{B-r}(x)}{1 - A(0)} \left[ \lambda^B + (1 - \lambda) \sum_{i=1}^B \lambda^{i-1} V_{B-i}(0, s) \right]. \quad (29)$$

Now we calculate  $a^k(x)$ ,  $A(0)$  in case when  $\delta \sim ge(\lambda)$ . Employing the formulae (9), (12) and performing the necessary calculations we find that

$$\begin{aligned} a^k(x) &= \tilde{f}_x(s) + (1 - \tilde{f}(s)) S_{k-1}^s(x) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \left[ \tilde{f}(s) + (1 - \tilde{f}(s)) \mathbf{E} S_{\delta+B-1}^s \right], \\ 1 - A(0) &= \frac{1 - \lambda}{\lambda \mathbf{E} Q_{\delta+B}^s} [1 - Q_0^s(0)] \left[ \tilde{f}(s) + (1 - \tilde{f}(s)) \mathbf{E} S_{\delta+B-1}^s \right], \\ \lambda^B + (1 - \lambda) \sum_{i=1}^B \lambda^{i-1} V_{B-i}(0, s) &= \frac{1 - \lambda}{\lambda \mathbf{E} Q_{\delta+B}^s} [1 - Q_0^s(0)]. \end{aligned}$$

Substituting the right-hand sides of these equalities into (29) and taking into account that  $\bar{T}_r^B(x) \sim ge(\lambda)$ , we derive (27).

The limiting formulae (19) were obtained in [26]. Similarly (see [26]) we derive for all  $k > 0$  that

$$\lim_{B \rightarrow \infty} B^{-2} S_{[kB]}^{s/B^2}(x) = \frac{1}{s \mathbf{E} \eta} \left( \text{ch} \left( k \sqrt{2s}/\sigma \right) - 1 \right) = \lim_{B \rightarrow \infty} B^{-2} \mathbf{E} S_{[kB]+\delta}^{s/B^2}. \quad (30)$$

Letting  $B \rightarrow \infty$ , we have  $\tilde{f}_x(s/B^2) = 1 - \mathbf{E} \eta_x s/B^2 + o(s/B^2)$ , which implies for  $r \in (0, 1)$  that

$$\lim_{B \rightarrow \infty} \mathbf{E} e^{-s\bar{\tau}_{[rB]}^B(x)/B^2} = \frac{1 + (\text{ch}(k\sqrt{2s}/\sigma) - 1)}{1 + (\text{ch}(\sqrt{2s}/\sigma) - 1)} = \frac{\text{ch}(k\sqrt{2s}/\sigma)}{\text{ch}(\sqrt{2s}/\sigma)}, \quad k = 1 - r.$$

The equality (28) follows from the following  $\mathbf{E} \bar{\tau}_r^B(x) = - \frac{d}{ds} \bar{v}_r^s(x) \Big|_{s=0}$ .  $\square$

### 3.2 Increments of the process reflected in its supremum

Define  $\bar{D}_0^k(x, t) = D_x(t) - \max \left\{ 0, \sup_{[0, t]} D_x(\cdot) - k \right\} \in \overline{-\infty, k}$ , the process reflected from the upper boundary  $k \in \mathbb{Z}^+$  generated by its supremum.

**Theorem 4.** Let  $\{\overline{D}_0^k(x, t)\}_{t \geq 0}$  be the process reflected from the upper boundary and  $\overline{p}_k^s(x, u) = \mathbf{P}[\overline{D}_0^k(x, \nu_s) \leq u]$ ,  $u \in \overline{-\infty, k}$  be the distribution of its increments on the exponential interval  $[0, \nu_s]$ . Then

- (i) for all  $k \in \mathbb{Z}^+$   $\overline{p}_k^s(x, k) = 1$ ; and for  $u \in \overline{-\infty, k-1}$ ,  $x \geq 0$  the following equality holds

$$\overline{p}_k^s(x, u) = A_x^u(s) + c(s)^{k-u-1} F(s) \left( \frac{\tilde{f}_x(s)}{1 - \tilde{f}(s)} + S_{k-1}^s(x) \right), \quad (31)$$

where  $A_x^u(s) = 0$ , for  $u < 0$ ,  $F(s) = s(1 - c(s))/(1 - \lambda)(s - k(c(s)))$ ;

- (ii) under the conditions (A) for  $k > 0$ ,  $u \leq k$  the following relation is valid

$$\lim_{B \rightarrow \infty} \mathbf{P}[\overline{D}_0^{[kB]}(x, tB^2) \leq [uB]] = 1 - \frac{1}{\sigma\sqrt{2\pi t}} \int_u^{2k-u} e^{-v^2/2\sigma^2 t} dv; \quad (32)$$

- (iii) if  $\rho > 1$ , then the ergodic distribution  $p_k(u) = \lim_{t \rightarrow \infty} \mathbf{P}[\overline{D}_0^k(x, t) \leq u]$  exists,

$$p_k(u) = \frac{\mathbf{E}\kappa}{\rho} \frac{1 - c}{1 - \mathbf{E}c\kappa} c^{k-u-1}, \quad u \in \overline{-\infty, k-1}, \quad c = \lim_{s \rightarrow 0} c(s) \in (\lambda, 1).$$

*Proof.* Define the generating function distribution of the process

$$\overline{P}_k^s(x, z) = \mathbf{E} z^{\overline{D}_0^k(x, \nu_s)} = \sum_{i=-\infty}^k z^i \mathbf{P}[\overline{D}_0^k(x, \nu_s) = i], \quad |z| \geq 1, \quad k \in \mathbb{Z}^+.$$

It is obvious that  $\overline{P}_k^s(x, 1) = \mathbf{P}[\overline{D}_0^k(x, \nu_s) \leq k] = 1$ . In accordance with the total probability law and the definition of the process  $\overline{D}_0^k(x, t)$  we can write

$$\begin{aligned} \overline{P}_k^s(x, z) &= E_k^+(x, z, s) + z^k \int_0^\infty f_x^k(dl)(1 - \tilde{f}_l(s)) \\ &\quad + z^k \int_0^\infty f_x^k(dl) \tilde{f}_l(s) (1 - \lambda) \sum_{i=1}^\infty \lambda^{i-1} z^{-i} \overline{P}_i^s(0, z), \end{aligned} \quad (33)$$

where the function  $E_k^+(x, z, s) = \mathbf{E}[e^{-zD_x(\nu_s)}; \tau^k(x) > \nu_s]$  is given by (14), and the function  $f_x^k(dl) = \mathbf{E}[e^{-s\tau^k(x)}; \eta^k(x) \in dl]$  is determined by (9). This equation means the following. The sample paths on which the increments of the process  $D_x(t)$  occur can be decomposed into three types: 1) the sample paths which do not intersect the upper boundary  $k$  (the first term of the right-hand side); 2) the sample paths which do intersect the upper boundary and stay there (the second term); 3) the sample paths cross the upper boundary and then they are reflected (the third term). After some calculation which we skip, one can see that the formula (9) implies that

$$\int_0^\infty f_x^k(dl) \tilde{f}_l(s) = \tilde{f}_x(s) + (1 - \tilde{f}(s)) S_k^s(x) - \frac{1 - \tilde{f}(s)}{1 - c(s)} Q_k^s(x).$$

Letting  $x = 0$  in (33) and taking into account the latter equality, we derive

$$(1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} z^{-i} \bar{P}_i^s(0, z) = \frac{F(s)}{1 - \tilde{f}(s)} \frac{1/z - 1}{1 - c(s)/z}, \quad |z| \geq 1.$$

Substituting the right-hand of this equality and the expression (14) for  $E_k^+(x, z, s)$  into (33), we find that ( $|z| \geq 1$ )

$$\bar{P}_k^s(x, z) = z^k + (1 - z) \sum_{i=0}^{k-1} z^i A_x^i(s) + z^k F(s) \frac{1/z - 1}{1 - c(s)/z} \left( \frac{\tilde{f}_x(s)}{1 - \tilde{f}(s)} + S_{k-1}^s(x) \right).$$

Comparing the coefficients of  $z^i$ ,  $i \in \{k, k-1, \dots\}$ , we get

$$\begin{aligned} \mathbf{P} \left[ \bar{D}_0^k(x, \nu_s) = k \right] &= 1 - A_x^{k-1}(s) - F(s) \left( \frac{\tilde{f}_x(s)}{1 - \tilde{f}(s)} + S_{k-1}^s(x) \right), \\ \mathbf{P} \left[ \bar{D}_0^k(x, \nu_s) = i \right] &= A_x^i(s) - A_x^{i-1}(s) + \\ &+ F(s) c(s)^{k-i-1} (1 - c(s)) \left( \frac{\tilde{f}_x(s)}{1 - \tilde{f}(s)} + S_{k-1}^s(x) \right), \quad i < k. \end{aligned}$$

One can see that the second formula implies the equality (31) of the theorem. Let us verify (32). It follows from the first formula of (19) that

$$F(s/B^2) = \frac{s}{B^2} \mathbf{E}\eta + o(B^{-2}), \quad \lim_{B \rightarrow \infty} c(s/B^2)^{[B(k-u)]-1} = e^{-(k-u)\sqrt{2s}/\sigma}.$$

Denote  $\tilde{p}_k^t(x, u, B) = \mathbf{P} \left[ \bar{D}_0^{[kB]}(x, tB^2) \leq [uB] \right]$ ,  $k > 0$ . Then (19), (30)

$$\begin{aligned} \lim_{B \rightarrow \infty} \int_0^\infty e^{-st} \tilde{p}_k^t(x, u, B) dt &= \frac{1}{s} \lim_{B \rightarrow \infty} \bar{p}_{[kB]}^{s/B^2}(x, [uB]) = \\ &= \frac{1}{s} \left( 1 - \text{ch} \left( u^+ \sqrt{2s}/\sigma \right) + e^{-(k-u)\sqrt{2s}/\sigma} \text{ch} \left( k \sqrt{2s}/\sigma \right) \right) = \\ &= \frac{1}{s} \mathbf{I}_{\{u < 0\}} \left( e^{u\sqrt{2s}/\sigma}/2 + e^{-(2k-u)\sqrt{2s}/\sigma}/2 \right) + \\ &+ \frac{1}{s} \mathbf{I}_{\{u \in [0, k]\}} \left( 1 - e^{-u\sqrt{2s}/\sigma}/2 + e^{-(2k-u)\sqrt{2s}/\sigma}/2 \right), \quad u \leq k, \end{aligned}$$

where  $u^+ = \max\{0, u\}$ . Employing the formula  $\frac{1}{s} e^{-a\sqrt{2s}/\sigma} = 2 \int_0^\infty e^{-st} \mathbf{P}[w_t \geq a] dt$ , to invert the Laplace transform, we derive the limiting equality of the theorem.

For  $\rho > 1$  the mathematical expectation of  $\tau^k(x)$  is finite. It follows from (10) that

$$\mathbf{E}\tau^k(x) = \frac{Q_k(x)}{(1 - \lambda)k(c)} + \sum_{i=0}^k \rho_i \left[ 1 - \frac{Q_{k-i}(x)}{1 - \lambda} \right] < \infty,$$

where  $\rho_i = \lim_{s \rightarrow 0} s^{-1} \tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}[\pi(t) = i] dt < \infty$ . Moreover, the process  $\bar{D}_0^k(x, t)$  is of regenerative type [33]. The instants if the passages of the upper boundary are the regeneration times. Hence, [33] there exists the ergodic distribution of the process  $p_k(u) = \lim_{t \rightarrow \infty} \mathbf{P} \left[ \bar{D}_0^k(x, t) \leq u \right]$ . To determine this distribution, it suffices to apply to (31) the Tauberian theorem:  $p_k(u) = \lim_{s \rightarrow 0} \bar{p}_k^s(x, u)$ .  $\square$

Let  $\{\overline{D}_0^k(x, t)\}_{t \geq 0}$  be the process reflected from the upper boundary. Define for  $r, k \in \mathbb{Z}^+$

$$\overline{\tau}_{r,k}(x) = \inf\{t : \overline{D}_0^k(x, t) < -r\} \stackrel{\text{def}}{=} \overline{\tau}, \quad \overline{T}_{r,k}(x) = -\overline{D}_0^k(x, \overline{\tau}) - r \stackrel{\text{def}}{=} \overline{T},$$

the first exit time from the interval  $[-r, k]$  by the process  $\overline{D}_0^k(x, t)$  and the value of the overshoot through the lower boundary  $-r$ . Since  $X_t$  is homogeneous with respect to the first component, then  $\{\overline{\tau}_{r,k}(x), \overline{T}_{r,k}(x)\}$  are identically distributed as  $\{\overline{\tau}_r^B(x), \overline{T}_r^B(x)\}$ ,  $B = k + r$  and their joint distribution is determined by (27).

**Theorem 5.** *Let  $\{\overline{D}_0^k(x, t)\}_{t \geq 0}$  be the process reflected from the upper boundary,  $\overline{p}_{r,k}^s(x, u) = \mathbf{P} \left[ \overline{D}_0^k(x, \nu_s) \leq u; \overline{\tau}_{r,k}(x) > \nu_s \right]$ ,  $u \in \overline{-r, k}$  be the distribution of the increments of the process on the interval  $[0, \nu_s]$  on the event  $\{\overline{\tau}_{r,k}(x) > \nu_s\}$ .*

(i) *the distribution of the increments is such that for all  $r, k \in \mathbb{Z}^+$ ,*

$$\begin{aligned} \overline{p}_{r,k}^s(x, k) &= 1 - \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E} S_{\delta+B-1}^s}, \quad B = k + r, \\ \overline{p}_{r,k}^s(x, u) &= A_x^u(s) - \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E} S_{\delta+B-1}^s} \mathbf{E} A_0^{\delta+u+r}(s), \quad u \in \overline{-r, k-1}; \end{aligned} \quad (34)$$

(ii) *under the conditions (A) the following limiting equality holds*

$$\begin{aligned} \lim_{B \rightarrow \infty} \mathbf{P} \left[ \overline{D}_0^{[kB]}(x, tB^2) \leq [uB]; \overline{\tau}_{[rB], [kB]}(x) > tB^2 \right] &\stackrel{\text{def}}{=} p(t) = \\ &= \frac{4}{\pi} \sum_{n \in \mathbb{Z}^+} \frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{n + \frac{1}{2}} \sin \left( r \left( n + \frac{1}{2} \right) \pi \right) \sin^2 \left( \frac{r+u}{2} \left( n + \frac{1}{2} \right) \pi \right), \end{aligned} \quad (35)$$

where  $r \in (0, 1)$ ,  $k = 1 - r$ ,  $u \in [-r, k]$ .

*Proof.* In accordance with the total probability law, homogeneity of the process  $X_t$  with respect to the first component, Markov property of  $\overline{\tau}_{r,k}(x)$  and the properties of the exponential variable  $\nu_s$  we can write

$$\overline{p}_k^s(x, u) = \overline{p}_{r,k}^s(x, u) + \overline{v}_r^s(x)(1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} \overline{p}_{i+B}^s(0, u + r + i), \quad u \in \overline{-\infty, k},$$

where the function  $\overline{p}_k^s(x, u) = \mathbf{P} \left[ \overline{D}_0^k(x, \nu_s) \leq u \right]$  is determined in Theorem 4.

This equation means that increments of the process  $\overline{D}_0^k(x, \nu_s)$  are realized either on the sample paths which do not exit the interval  $[-r, k]$ , or on the sample paths which do exit the interval and the further evaluation of the process is its probabilistic replica on  $[0, \nu_s]$ . Substituting the expression for the function  $\overline{p}_k^s(x, u)$  into (31) after necessary calculations we derive (34).

For  $r \in (0, 1)$ ,  $k = 1 - r$ ,  $u \in [-r, k]$  denote

$$\overline{p}_{r,k}^t(x, u, B) = \mathbf{P} \left[ \overline{D}_0^{[kB]}(x, tB^2) \leq [uB]; \overline{\tau}_{[rB], [kB]}(x) > tB^2 \right].$$

Employing the third formula of (19), the limiting equality of Theorem 3, we find

$$\begin{aligned} \frac{1}{s} \lim_{B \rightarrow \infty} \bar{p}_{[kB]}^{s/B^2}(x, [uB]) &= \lim_{B \rightarrow \infty} \int_0^\infty e^{-st} \bar{p}_{r,k}^t(x, u, B) dt = \\ &= \frac{1 - \text{ch}(u^+ \sqrt{2s}/\sigma)}{s} + \frac{1}{s} \frac{\text{ch}(k \sqrt{2s}/\sigma)}{\text{ch}(\sqrt{2s}/\sigma)} \left( \text{ch}((u+r) \sqrt{2s}/\sigma) - 1 \right) \stackrel{\text{def}}{=} p^*(s), \end{aligned} \quad (36)$$

where  $u^+ = \max\{0, u\}$ . When  $u \in [-r, 0]$  we derive from this formula that

$$p^*(s) = \frac{2}{s} \frac{\text{ch}(k \sqrt{2s}/\sigma)}{\text{ch}(\sqrt{2s}/\sigma)} \text{sh}^2 \left( \frac{r+u}{2} \sqrt{2s}/\sigma \right), \quad u \in [-r, 0].$$

It is clear that  $s = 0$  is not a singular point (pole or point of branching) of the function  $p^*(s)$ . In the semi-plane  $\Re(s) < 0$  this function has simple poles in

$$s_n = -\frac{1}{2} \sigma^2 \pi^2 \left( n + \frac{1}{2} \right)^2, \quad n \in \mathbb{Z}^+,$$

and it is analytic in the whole plane apart from these points. Hence, for  $\alpha > 0$

$$p(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} p^*(s) ds = \sum_{n \in \mathbb{Z}^+} \text{Res}_{s=s_n} p^*(s).$$

Calculating the residues of the function  $p^*(s)$  in  $s_n$ , we obtain the right-hand side of the formula (35) for  $u \in [-r, 0]$ . One can see that the first term in the right-hand side of (36) is analytic in the whole plane for  $u \in (0, k]$ . Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (36) is the same also for  $u \in [-r, 0]$ . Thus, the formula (35) holds for  $u \in [-r, k]$ .  $\square$

## 4 Applications for $M^\varkappa |G^\delta| 1|B$ system

Let  $B \in \mathbb{Z}^+$ ,  $r \in \overline{0, B+1}$ ,  $x \geq 0$ . Introduce the two-component Markov process

$$Y_{r,x}(t) = \{d_{r,x}(t), \eta_{r,x}(t)\} \in \overline{0, B+1} \times \mathbb{R}_+, \quad Y_{r,x}(0) = (r, x)$$

by means of the following recurrent equations

$$Y_{r,x}(t) = \begin{cases} \left( \overline{D}_r^{B+1}(x, t), \eta_x^+(t) \right), & 0 \leq t < \tilde{\tau}_r^{B+1}(x), \\ Y_{0,0}(t - \tilde{\tau}_r^{B+1}(x)), & t \geq \tilde{\tau}_r^{B+1}(x), \end{cases}, \quad r \in \overline{1, B+1}, \quad (37)$$

$$Y_{0,0}(t) = \begin{cases} (0, 0), & 0 \leq t < \tilde{\mu} \sim \exp(\mu), \\ Y_{r,0}(t - \tilde{\mu}) : \mathbf{P}[\varkappa = r], & r = \overline{1, B}, \quad t \geq \tilde{\mu}, \\ Y_{B+1,0}(t - \tilde{\mu}) : \mathbf{P}[\varkappa \geq B+1], & t \geq \tilde{\mu}, \end{cases} \quad (38)$$

where  $\tilde{\tau}_r^{B+1}(x) = \inf\{t : \overline{D}_r^{B+1}(x, t) < 1\}$ ,  $r = \overline{1, B+1}$ .

**Remark 1.** Since the process  $X_t$  is homogeneous with respect to the first component, then the random variable  $\tilde{\tau}_r^{B+1}(x)$  is identically distributed as  $\bar{\tau}_{r-1}^B(x)$  (27) and, hence,

$$\tilde{v}_r^s(x) = \mathbf{E} \left[ e^{-s\tilde{\tau}_r^{B+1}(x)}; \tilde{\tau}_r^{B+1}(x) < \infty \right] = \bar{v}_{r-1}^s(x), \quad r \in \overline{1, B+1}.$$

The process  $Y_{r,x}(t)_{\{t \geq 0\}}$  serves as a mathematical model of the functioning of  $M^\kappa | G^\delta | 1 | B$ ,  $(\delta \sim ge(\lambda))$  system, which has the following properties

- (i) The customers arrive in groups (batch arrivals) according to the Poisson process with intensity  $\mu > 0$ . The number of the customers in every group is represented by the random variable  $\kappa \in \mathbb{N}$ .
- (ii) The system has a finite waiting room (buffer) whose size is equal to  $B + 1 < \infty$ . Suppose that upon the arrival of a new claim of size  $\kappa$  it finds  $r \in \overline{0, B+1}$  occupied space in the waiting room. Then  $\min\{k, \kappa\}$  joins the queue, and loss of size  $\max\{0, \kappa - k\}$  occurs, where  $k = B + 1 - r$  is the size of empty space in the waiting room;
- (iii) The duration of service completion is arbitrary distributed as  $\eta > 0$ . Suppose, that at time  $t$  the service cycle is accomplished. Then the occupied space in the buffer is reduced by  $\min\{r, \delta\}$ , where  $r \in \overline{1, B+1}$  is the value of occupied space in the waiting room at time  $t - 0$ . If at the instant of the service completion  $r - \min\{r, \delta\} > 0$ , then a new service cycle starts. If at the instant of the service completion  $r - \min\{r, \delta\} = 0$ , then the new service cycle starts upon arrival of a new claim (after exponential time with parameter  $\mu > 0$ ).

For all  $t \geq 0$  the event  $\{Y_{r,x}(t) = (i, y)\}$ ,  $i \in \overline{1, B+1}$ ,  $y \geq 0$  means that at time  $t$  there are  $i$  occupied places in the waiting room, and  $y$  stands for time elapsed since the beginning of the service cycle. Here  $(r, x)$  is an initial state of the system.

The event  $\{Y_{r,x}(t) = (0, 0)\}$  means that at time  $t$  the waiting room is empty and the server is idle. The system stays in the  $(0, 0)$  state for an exponential period of time (with parameter  $\mu$ .)

Therefore,  $d_{r,x}(t)$  is the number of the customers in the waiting room at time  $t$ . If  $d_{r,x}(t) > 0$ , then  $\eta_{r,x}(t)$  is the time elapsed since the last start of the service cycle up to time  $t$ . If  $d_{r,x}(t) = 0$ , then  $\mathbf{P}[\eta_{r,x}(t) = 0] = 1$ .

#### 4.1 Busy period of the system

Assume that at time  $t_0 = 0$  system is in the state  $(r, x)$ , where  $r \in \overline{1, B+1}$  is the number of the customers in the waiting room, and  $x \geq 0$  is the duration of the current service cycle. Introduce the random variable

$$b_r(x) = \inf\{t : d_{r,x}(t) = 0\}$$

i.e. the instant at which the system for the first time becomes empty. Thus, the interval  $[0, b_r(x)]$  is a busy period of  $(r, x)$  type.

**Corollary 1.** Let  $b_r^s(x) = \mathbf{E} [e^{-sb_r(x)}; b_r(x) < \infty]$  be the Laplace transform of the busy period of type  $(r, x)$ . Then the following relation holds

$$b_r^s(x) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{B-r}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E} S_{\delta+B-1}^s}, \quad x \geq 0, \quad r \in \overline{1, B+1}, \quad (39)$$

where

$$S_k^s(x) = \sum_{i=0}^k Q_i^s(x), \quad \mathbf{E} S_{\delta+B-1}^s = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} S_{i-1+B}^s(0).$$

Observe, that the random variable  $b_r(x)$  is proper ( $\mathbf{P}[b_r(x) < \infty] = 1$ ) and

$$\mathbf{E} b_r(x) = \mathbf{E} \eta_x - \mathbf{E} \eta + \mathbf{E} \eta [\mathbf{E} S_{\delta+B-1} - S_{B-r}(x)] < \infty, \quad (40)$$

where  $S_k(x) = S_k^0(x)$ ,  $\mathbf{E} S_{\delta+B} = \mathbf{E} S_{\delta+B}^0$ .

These formulae follow straightforwardly from Theorem 3 and Remark 1.

## 4.2 Time of the first loss of a customer

Suppose that the system starts functioning from the state  $(r, x)$  and denote by  $l_r(x)$  the time of the first loss of the customer (a group of customers).

**Corollary 2.** Let  $l_r^s(x) = \mathbf{E} [e^{-sl_r(x)}; l_r(x) < \infty]$  be the Laplace transform of  $l_r(x)$ . Then the following relation is valid

$$\begin{aligned} l_0^s(0) &= 1 - \mathbf{E} A_0^{\delta+B}(s) + \mathbf{E} Q_{\delta+B}^s \frac{\mathbf{E} A_0^{\delta+B}(s) - \frac{\mu}{s+\mu} \tilde{A}(s) - \frac{s}{s+\mu}}{\mathbf{E} Q_{\delta+B}^s - \frac{\mu}{s+\mu} \tilde{Q}(s)}, \\ l_r^s(x) &= 1 - A_x^k(s) + Q_k^s(x) \frac{\mathbf{E} A_0^{\delta+B}(s) - \frac{\mu}{s+\mu} \tilde{A}(s) - \frac{s}{s+\mu}}{\mathbf{E} Q_{\delta+B}^s - \frac{\mu}{s+\mu} \tilde{Q}(s)}, \end{aligned} \quad (41)$$

where  $r \in \overline{1, B+1}$ ,  $k = B+1-r$ ,

$$\tilde{Q}(s) = \sum_{i=1}^{B+1} \mathbf{P}[\mathcal{X} = i] Q_{B+1-i}^s, \quad \tilde{A}(s) = \sum_{i=1}^{B+1} \mathbf{P}[\mathcal{X} = i] A_0^{B+1-i}(s).$$

Note, that the random variables  $l_0(0)$ ,  $l_r(x)$  are proper, and they have finite mathematical expectations.

*Proof.* Let  $r \in \overline{1, B+1}$ ,  $x \geq 0$ . Denote by  $\tilde{\chi}_r^{B+1}(x) = \inf\{t : r + D_x(t) \notin [1, B+1]\}$  the first exit time from the interval  $[1, B+1]$  by the process  $r + D_x(t)$ . Since the process  $X_t$  is homogeneous with respect to the first component, then the random variable  $\tilde{\chi}_r^{B+1}(x)$  is identically distributed as  $\chi_{r-1}^B(x)$  and its Laplace transforms are determined by the formulae of Theorem 1. In accordance with

the definition of the process  $Y_{r,x}(t)$  we can write the following system of the equations for the functions  $l_r^s(x)$ ,  $l_0^s(0)$

$$\begin{aligned} l_r^s(x) &= V^{B+1-r}(x, s) + V_{r-1}(x, s) l_0^s(0), \quad r \in \overline{1, B+1}, \quad x \geq 0, \\ l_0^s(0) &= \frac{\mu}{s + \mu} \hat{a}_{B+1} + \frac{\mu}{s + \mu} \sum_{i=1}^{B+1} a_i l_i^s(0), \end{aligned} \quad (42)$$

where  $a_i = \mathbf{P}[\varkappa = i]$ ,  $\hat{a}_i = \mathbf{P}[\varkappa > i]$ , and (13)

$$V_{r-1}(x, s) = \frac{Q_{B+1-r}^s(x)}{\mathbf{E} Q_{\delta+B}^s}, \quad (43)$$

$$V^{B+1-r}(x, s) = 1 - A_x^{B+1-r}(s) - \frac{Q_{B+1-r}^s(x)}{\mathbf{E} Q_{\delta+B}^s} \left( 1 - \mathbf{E} A_0^{\delta+B}(s) \right),$$

Substituting the right-hand side of the first equation of (42) for  $x = 0$  into the second one, we get

$$l_0^s(0) = \frac{\mu}{s + \mu} \hat{a}_{B+1} + \frac{\mu}{s + \mu} \sum_{i=1}^{B+1} a_i V^{B+1-i}(0, s) + \frac{\mu}{s + \mu} \sum_{i=1}^{B+1} a_i V_{i-1}(0, s) l_0^s(0).$$

The latter equation yields

$$l_0^s(0) = \frac{\mu}{s + \mu} \left( \hat{a}_{B+1} + \sum_{i=1}^{B+1} a_i V^{B+1-i}(0, s) \right) \left( 1 - \frac{\mu}{s + \mu} \sum_{i=1}^{B+1} a_i V_{i-1}(0, s) \right)^{-1}.$$

Taking into account the first formula of (42) and (43), we derive the equalities (41) of the corollary.  $\square$

### 4.3 Number of the customers in the system

Let  $\nu_s \sim \exp(s)$  be the exponential random variable with parameter  $s > 0$ . Introduce the transient probabilities of the process  $d_{r,x}(t)_{\{t \geq 0\}}$  :

$$\begin{aligned} q_{r,x}^s(u) &= \mathbf{P}[d_{r,x}(\nu_s) \leq u], \quad q_{0,0}^s(u) = \mathbf{P}[d_{0,0}(\nu_s) \leq u], \quad r, u \in \overline{1, B+1}, \\ q_{r,x}^s(0) &= \mathbf{P}[d_{r,x}(\nu_s) = 0], \quad q_{0,0}^s(0) = \mathbf{P}[d_{0,0}(\nu_s) = 0]. \end{aligned}$$

Denote  $\tilde{b}(s) = \hat{a}_B b_{B+1}^s(0) + \sum_{i=1}^B a_i b_i^s(0)$ .

**Theorem 6.** *The distribution of the number of the customers at time  $\nu_s$  is such that*

$$\begin{aligned} q_{0,0}^s(u) &= \mathbf{E} A_0^{\delta+u-1}(s) + \frac{C_u(s, \lambda)}{s + \mu - \mu \tilde{b}(s)}, \quad q_{0,0}^s(B+1) = 1 - \frac{s}{s + \mu - \mu \tilde{b}(s)}, \\ q_{r,x}^s(u) &= A_x^{u-r}(s) + b_r^s(x) \frac{C_u(s, \lambda)}{s + \mu - \mu \tilde{b}(s)}, \quad q_{r,x}^s(B+1) = 1 - \frac{s b_r^s(x)}{s + \mu - \mu \tilde{b}(s)}, \\ q_{0,0}^s(0) &= \frac{s}{s + \mu - \mu \tilde{b}(s)}, \quad q_{r,x}^s(0) = \frac{s b_r^s(x)}{s + \mu - \mu \tilde{b}(s)}, \end{aligned}$$

where

$$C_u(s, \lambda) = \frac{s Q_u^s}{1 - \lambda} - s + \lambda(s + \mu) \left( A_0^u(s) - \mathbf{E} A_0^{\delta+u}(s) \right).$$



**Corollary 3.** Let  $\pi_i = \lim_{t \rightarrow \infty} \mathbf{P}[d_{(\cdot)}(t) = i]$ ,  $i \in \overline{0, B+1}$  be the stationary distribution of the number of customers in the system  $M^\infty|G^\delta|1|B$ . Then

$$\pi_0 = \left[ 1 + \mu \mathbf{E} \eta \left( \frac{\lambda}{1-\lambda} \mathbf{E} Q_{\delta+B} + \sum_{i=0}^B \hat{a}_i Q_{B-i} \right) \right]^{-1},$$

$$\pi_{B+1} = 1 - \pi_0(1 + C_B(\lambda)), \quad \pi_i = \pi_0(C_i(\lambda) - C_{i-1}(\lambda)), \quad i \in \overline{1, B},$$

where  $\rho_i = \lim_{s \rightarrow 0} s^{-1} \tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}[\pi(t) = i] dt < \infty$ ,  $C_0(\lambda) = 0$ ,

$$C_u(\lambda) = \frac{Q_u}{1-\lambda} - 1 + \lambda \mu \left( A_0^u - \mathbf{E} A_0^{\delta+u} \right), \quad A_0^u = \lim_{s \rightarrow 0} \frac{1}{s} A_0^u(s) = \sum_{i=0}^u \rho_i \left[ 1 - \frac{Q_{u-i}}{1-\lambda} \right].$$

*Proof.* By  $\tilde{p}_{r,x}^s(u) = \mathbf{P}[d_{r,x}(\nu_s) \leq u; b_r(x) > \nu_s]$ ,  $r, u \in \overline{1, B+1}$  denote the transient probability of the process  $d_{r,x}(\nu_s)$  on the event  $\{b_r(x) > \nu_s\}$ . Taking into account the homogeneity of the process  $X_t$  with respect to the first component, the definition of the process  $d_{r,x}(t)$  and the formulae (34) of Theorem 5, we derive

$$\tilde{p}_{r,x}^s(B+1) = 1 - b_r^s(x), \quad \tilde{p}_{r,x}^s(u) = A_x^{u-r}(s) - b_r^s(x) \mathbf{E} A_0^{\delta+u-1}(s), \quad u \in \overline{1, B},$$

where the function  $b_r^s(x)$  is determined by (39).

In accordance with the definition of the process  $Y_{r,x}(t)$  we can write the following equations for the functions  $q_{r,x}^s(u)$ ,  $q_{0,0}^s(u)$  for  $u \in \overline{1, B}$

$$q_{r,x}^s(u) = \tilde{p}_{r,x}^s(u) + b_r^s(x) q_{0,0}^s(u),$$

$$q_{0,0}^s(u) = \frac{\mu}{s + \mu} \left[ \hat{a}_B q_{B+1,0}^s(u) + \sum_{i=1}^B a_i q_{i,0}^s(u) \right]. \quad (44)$$

Substituting the right-hand side of the second equation into the first one, we get

$$q_{r,x}^s(u) = \tilde{p}_{r,x}^s(u) + b_r^s(x) \frac{\mu}{s + \mu} \tilde{q}(s, u),$$

where  $\tilde{q}(s, u) = \hat{a}_B q_{B+1,0}^s(u) + \sum_{i=1}^B a_i q_{i,0}^s(u)$ . Letting  $x = 0$  in the latter equality implies that

$$\hat{a}_B q_{B+1,0}^s(u) = \hat{a}_B \tilde{p}_{B+1,0}^s(u) + \hat{a}_B b_{B+1}^s(0) \frac{\mu}{s + \mu} \tilde{q}(s, u),$$

$$\sum_{i=1}^B a_i q_{i,0}^s(u) = \sum_{i=1}^B a_i \tilde{p}_{i,0}^s(u) + \sum_{i=1}^B a_i b_i^s(0) \frac{\mu}{s + \mu} \tilde{q}(s, u).$$

Adding these equalities, we obtain the function  $\tilde{q}(s, u)$

$$\tilde{q}(s, u) = \tilde{p}(s, u) \left( 1 - \frac{\mu}{s + \mu} \tilde{b}(s) \right)^{-1}, \quad u \in \overline{1, B},$$

where  $\tilde{b}(s) = \hat{a}_B b_{B+1}^s(0) + \sum_{i=1}^B a_i b_i^s(0)$ ,

$$\tilde{p}(s, u) = \hat{a}_B \tilde{p}_{B+1,0}^s(u) + \sum_{i=1}^B a_i \tilde{p}_{i,0}^s(u) = \sum_{i=1}^u a_i A_0^{u-i}(s) - \tilde{b}(s) \mathbf{E} A_0^{\delta+u-1}(s).$$

Substituting the expression for the function  $\tilde{q}(s, u)$  into (44), we find that

$$\begin{aligned} q_{0,0}^s(u) &= \mathbf{E} A_0^{\delta+u-1}(s) + \frac{C_u(s, \lambda)}{s + \mu - \mu \tilde{b}(s)}, \\ q_{r,x}^s(u) &= A_x^{u-r}(s) + b_r^s(x) \frac{C_u(s, \lambda)}{s + \mu - \mu \tilde{b}(s)}, \end{aligned} \quad (45)$$

where

$$C_u(s, \lambda) = \frac{s Q_u^s}{1 - \lambda} - s + \lambda(s + \mu) \left( A_0^u(s) - \mathbf{E} A_0^{\delta+u}(s) \right).$$

To derive the latter equalities, we used the following relation

$$\mu \sum_{i=1}^u a_i A_0^{u-i}(s) - (s + \mu) A_0^u(s) = \frac{s}{1 - \lambda} Q_u^s - s, \quad u \in \overline{1, \infty},$$

which follows from the definition of the function  $A_x^u(s)$ . If  $u = B + 1$ , then  $\tilde{p}_{r,x}^s(B + 1) = 1 - b_r^s(x)$ ,  $\tilde{p}(s, B + 1) = 1 - \tilde{b}(s)$  and the following formulae are valid

$$q_{0,0}^s(B + 1) = 1 - \frac{s}{s + \mu - \mu \tilde{b}(s)}, \quad q_{r,x}^s(B + 1) = 1 - \frac{s b_r^s(x)}{s + \mu - \mu \tilde{b}(s)}. \quad (46)$$

Taking into account the definition of the process  $Y_{r,x}(t)$ , we can write the following equations for the functions  $q_{r,x}^s(0)$ ,  $q_{0,0}^s(0)$

$$\begin{aligned} q_{r,x}^s(0) &= b_r^s(x) q_{0,0}^s(0), \\ q_{0,0}^s(0) &= \frac{s}{s + \mu} + \frac{\mu}{s + \mu} \left[ \hat{a}_B q_{B+1,0}^s(0) + \sum_{i=1}^B a_i q_{i,0}^s(0) \right]. \end{aligned}$$

Solving this system yields

$$q_{0,0}^s(0) = \frac{s}{s + \mu - \mu \tilde{b}(s)}, \quad q_{r,x}^s(0) = \frac{s b_r^s(x)}{s + \mu - \mu \tilde{b}(s)}. \quad (47)$$

Observe that  $\lim_{s \rightarrow 0} A_x^u(s) = \lim_{s \rightarrow 0} \mathbf{E} A_0^{\delta+u}(s) = 0$ ,  $\lim_{s \rightarrow 0} b_r^s(x) = 1$ . It follows from (45)–(47) and properties of the Laplace transforms that

$$\begin{aligned} \lim_{s \rightarrow 0} q_{r,x}^s(u) &= \lim_{s \rightarrow 0} q_{0,0}^s(u) = q(u) = \lim_{t \rightarrow \infty} \mathbf{P}[d_{(\cdot)}(t) \leq u], \quad u \in \overline{1, B + 1}, \\ \lim_{s \rightarrow 0} q_{r,x}^s(0) &= \lim_{s \rightarrow 0} q_{0,0}^s(0) = q(0) = \lim_{t \rightarrow \infty} \mathbf{P}[d_{(\cdot)}(t) = 0]. \end{aligned}$$

The formulae (47) imply that

$$q(0) = \lim_{s \rightarrow 0} \frac{s}{s + \mu - \mu \tilde{b}(s)} = \left[ 1 + \mu \mathbf{E} \eta \left( \frac{\lambda}{1 - \lambda} \mathbf{E} Q_{\delta+B} + \sum_{i=0}^B \hat{a}_i Q_{B-i} \right) \right]^{-1}.$$

In view of (45), (46) we find

$$q(B+1) = 1 - q(0), \quad q(u) = q(0)C_u(\lambda), \quad u \in \overline{1, B},$$

where  $\rho_i = \lim_{s \rightarrow 0} s^{-1} \tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}[\pi(t) = i] dt < \infty$ ,

$$C_u(\lambda) = \frac{Q_u}{1-\lambda} - 1 + \lambda \mu \left( A_0^u - \mathbf{E} A_0^{\delta+u} \right), \quad A_0^u = \lim_{s \rightarrow 0} \frac{1}{s} A_0^u(s) = \sum_{i=0}^u \rho_i \left[ 1 - \frac{Q_{u-i}}{1-\lambda} \right].$$

□

#### 4.4 $M^\varkappa|G|1|B$ system

Let us stress the following fact. If we set the parameter  $\lambda = 0$  in the geometrical distribution  $\mathbf{P}[\delta = n] = (1 - \lambda)\lambda^{n-1}$ ,  $n \in \mathbb{N}$ ,  $\lambda \in [0, 1)$  of the random variable  $\delta$ , then  $\mathbf{P}[\delta = 1] = 1$ . In other words it means that the process  $\{D_x(t)\}_{t \geq 0}$  has unit negative jumps at the times instants  $\{\eta_n(x)\}_{n \in \mathbb{N}}$  and  $\delta_{N_x(t)} = N_x(t)$ . Then, it follows from (2) that

$$D_x(t) = \pi(t) - N_x(t) \in \mathbb{Z}, \quad t \geq 0. \quad (48)$$

We will call this process a difference of the compound Poisson process and a simple renewal process. Setting the parameter  $\lambda = 0$  in the statements of Lemma 1 leads to the following result.

**Lemma 5.** *For  $s > 0$  the equation  $\theta = \tilde{f}(s - k(\theta))$  has a unique solution  $c(s)$  inside the circle  $|\theta| < 1$ . This solution is positive,  $c(s) \in (0, 1)$ . If  $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$ ,  $\rho = \mu \mathbf{E}[\varkappa] \mathbf{E}[\eta]$ , then for  $\rho > 1$ ,  $\lim_{s \rightarrow 0} c(s) = c \in (0, 1)$ ; and for  $\rho \leq 1$ ,  $\lim_{s \rightarrow 0} c(s) = 1$ .*

The statements of Lemma's 2–4 and Theorem's 1–6 can be reformulated in a similar way. Letting  $\lambda = 0$  in the defining formula (8) for all  $s, x \geq 0$  we get

$$Q_k^s(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{1}{\theta^{k+1}} \frac{\tilde{f}_x(s - k(\theta))}{\tilde{f}(s - k(\theta)) - \theta} d\theta, \quad \alpha \in (0, c(s)) \quad (49)$$

the resolvent sequence of the process  $\{D_x(t)\}_{t \geq 0}$ , which is given by (48). This resolvent sequence has been introduced in [23]. Setting  $\lambda = 0$  in (9), (10), we obtain

$$\begin{aligned} f_x^k(dl, m, s) &= e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x)) \\ &\quad + \Phi_0^s(dl, m) Q_k^s(x) - e^{-sl} [1 - F(l)] \sum_{i=0}^k Q_i^s(x) p_{k-i}^m(dl), \\ \mathbf{E} e^{-s\tau^k(x)} &= 1 - \frac{s}{s - k(c(s))} Q_k^s(x) - A_x^k(s) \end{aligned} \quad (50)$$

the Laplace transforms of the upper one-boundary functionals of the process  $\{D_x(t)\}_{t \geq 0}$  (48), where  $\tilde{\rho}_i(s) = s \int_0^\infty e^{-st} \mathbf{P}[\pi(t) = i] dt$ ,

$$A_x^k(s) = \sum_{i=0}^k \tilde{\rho}_i(s) [1 - Q_{k-i}^s(x)], \quad \Phi_0^s(dl, m) = e^{-sl} [1 - F(l)] \sum_{k \in \mathbb{Z}^+} c(s)^k p_k^m(dl).$$

We have introduced the auxiliary functions and the resolvent sequence of the process (48), therefore we can state the following result.

**Corollary 4.** *Let  $\{D_x(t)\}_{t \geq 0}$  be the difference of the compound Poisson process and the renewal process (48),  $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$ ,  $x \geq 0$  be the resolvent sequence of the process given by (49),  $Q_k^s \stackrel{\text{def}}{=} Q_k^s(0)$ . Then*

- (i) *the Laplace transforms  $V_r^x(m, s)$ ,  $V_x^k(dl, m, s)$  of the joint distribution of  $\{\chi, L, T\}$  satisfy the following equalities for all  $x, s \geq 0$ ,  $m \in \mathbb{N}$*

$$V_r(x, i, s) = \frac{Q_k^s(x)}{Q_{B+1}^s} \delta_{i1}, \quad V^k(x, dl, i, s) = f^k(x, dl, i, s) - \frac{Q_k^s(x)}{Q_{B+1}^s} f^{B+1}(0, dl, i, s),$$

where  $\delta_{ij}$  is the Kronecker symbol and  $f^k(x, dl, m, s)$  is given by (50);

- (ii) *for the Laplace transforms of the first exit time  $\chi$  from the interval by the process  $\{D_x(t)\}_{t \geq 0}$  the formulae hold*

$$\mathbf{E}[e^{-s\chi}; \mathfrak{A}_r] = \frac{Q_k^s(x)}{Q_{B+1}^s}, \quad \mathbf{E}[e^{-s\chi}; \mathfrak{A}^k] = 1 - A_x^k(s) + \frac{Q_k^s(x)}{Q_{B+1}^s} (1 - A_0^{B+1}(s));$$

- (iii) *the exit probabilities from the interval by the process  $\{D_x(t)\}_{t \geq 0}$  satisfy the equalities*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{Q_{B+1}}, \quad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{Q_{B+1}},$$

where the resolvent sequence of the process  $\{Q_k(x)\}_{k \in \mathbb{Z}^+}$ ,  $x \geq 0$ ,  $Q_k \stackrel{\text{def}}{=} Q_k(0)$  is given by (49) for  $s = 0$ .

In order to prove the corollary, one has to put  $\lambda = 0$  in the statements of Theorem 1. The results obtained in Corollary 4 can be applied for studying the queueing systems  $M^\infty|G|1|B$  ( $\mathbf{P}[\delta = 1] = 1$ ) with finite waiting room. To illustrate this, we now will determine the distribution of the busy period, the number of the customers, time of the first loss of the customer and the number of lost customers at time of the first loss.

**Corollary 5.** *Let  $\mathbf{P}[\delta = 1] = 1$ ,  $b_r^s(x) = \mathbf{E}[e^{-sb_r(x)}; b_r(x) < \infty]$  be the Laplace transform of the busy period of type  $(r, x)$  of  $M^\infty|G|1|B$  system. Then the following relation is valid*

$$b_r^s(x) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{B-r}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))S_B^s}, \quad x \geq 0, \quad r \in \overline{1, B+1},$$

where  $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$ ,  $S_k^s(x) = 0$ , for  $k < 0$ . The random variable  $b_r(x)$  is proper and

$$\mathbf{E}b_r(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta[S_B - S_{B-r}(x)] < \infty,$$

where  $S_k(x) = S_k^0(x)$ .

These formulae were derived in [23]. To prove the corollary, it suffices to set  $\lambda = 0$  in the equalities of Corollary 3.

**Corollary 6.** *Let  $\mathbf{P}[\delta = 1] = 1$ ,  $l_r(x)$  be the time of the first loss of the batch of the customers in the system  $M^\varkappa|G|1|B$ , and  $l_r^s(x) = \mathbf{E}[e^{-sl_r(x)}; l_r(x) < \infty]$  be the Laplace transform of  $l_r(x)$ . Then the following formula holds*

$$l_r^s(x) = 1 - A_x^k(s) - Q_k^s(x) \frac{s}{s + \mu} \left( 1 - \frac{\mu}{s + \mu} \tilde{Q}(s)/Q_{B+1}^s \right)^{-1},$$

where  $r \in \overline{0, B+1}$ ,  $k = B+1-r$ ,  $\tilde{Q}(s) = \sum_{i=1}^{B+1} \mathbf{P}[\varkappa = i] Q_{B+1-i}^s$ . The random variable  $l_r(x)$  is proper and has finite mathematical expectation.

The function  $l_r^s(x)$  was found in [23] in a different form.

**Corollary 7.** *The distribution of the number of the customers in the system  $M^\varkappa|G|1|B$  at time  $\nu_s$  is such that for  $r \in \overline{0, B+1}$ ,  $u \in \overline{1, B+1}$*

$$q_{r,x}^s(u) = A_x^{u-r}(s) + b_r^s(x) \frac{s(Q_u^s - 1)}{s + \mu - \mu \tilde{b}(s)}, \quad q_{r,x}^s(B+1) = 1 - \frac{sb_r^s(x)}{s + \mu - \mu \tilde{b}(s)},$$

$$q_{r,x}^s(0,0) = \frac{s b_r^s(x)}{s + \mu - \mu \tilde{b}(s)}, \quad b_0^s(x) \stackrel{\text{def}}{=} 1.$$

**Corollary 8.** *Let  $\pi_i = \lim_{t \rightarrow \infty} \mathbf{P}[d_{(\cdot)}(t) = i]$ ,  $i \in \overline{0, B+1}$  be the stationary distribution of the number of the customers in the system  $M^\varkappa|G|1|B$ . Then*

$$\pi_0 = \left( 1 + \mu \mathbf{E}\eta \sum_{i=0}^B \hat{a}_i Q_{B-i} \right)^{-1},$$

$$\pi_{B+1} = 1 - \pi_0 Q_B, \quad \pi_i = \pi_0 (Q_i - Q_{i-1}), \quad i \in \overline{1, B},$$

where the resolvent sequence of the process  $\{Q_k(x)\}_{k \in \mathbb{Z}^+}$ ,  $Q_k \stackrel{\text{def}}{=} Q_k(0)$ ,  $Q_0 = 1$  is given by (49) for  $s = 0$ .

To prove Corollaries 6–8, it suffices to set  $\lambda = 0$  in the equalities of Corollaries 2, 3 and Theorem 6. Note, that the formulae of Corollary 8 were obtained in [23].

Suppose that the system starts functioning from the state  $(r, x)$ ,  $r \in \overline{0, B+1}$ . Denote by  $i_{r,x}$  the number of the lost customers at time of the first loss  $l_r(x)$ .

**Corollary 9.** *The generating function  $L_{r,x}^s(z) = \mathbf{E} [e^{-sl_r(x)} z^{i_{r,x}}]$  of the joint distribution  $\{l_r(x), i_{r,x}\}$  is such that*

$$L_{r,x}^s(z) = \frac{\mu}{s} \sum_{i=0}^k E^i(z) [A_x^{k-i}(s) - A_x^{k-i-1}(s)] + \\ + \mu \frac{Q_k^s(x)}{Q_{B+1}^s} \frac{\sum_{i=0}^{B+1} E^i(z) [Q_{B+1-i}^s - Q_{B-i}^s]}{s + \mu - \mu \tilde{Q}(s)/Q_{B+1}^s}, \quad k = B+1-r, \quad (51)$$

$$\mathbf{E} [e^{-sl_r(x)}; i_{r,x} = n] = \frac{\mu}{s} \left[ a_n A_x^k(s) + \sum_{i=1}^k (a_{n+i} - a_{n+i-1}) A_x^{k-i}(s) \right] + \\ + \mu \frac{Q_k^s(x)}{Q_{B+1}^s} \frac{a_n Q_{B+1}^s + \sum_{i=1}^{B+1} (a_{n+i} - a_{n+i-1}) Q_{B+1-i}^s}{s + \mu - \mu \tilde{Q}(s)/Q_{B+1}^s}, \quad n \in \mathbb{N},$$

where  $E^i(z) = \mathbf{E} [z^{\varkappa-i}; \varkappa > i]$ ,  $i \in \mathbb{Z}^+$ .

*Proof.* In accordance with the total probability law we can write the following system of equations for  $L_{r,x}^s(z)$

$$L_{r,x}^s(z) = \tilde{V}^k(x, z, s) + V_{r-1}(x, s) L_{0,0}^s(z), \quad r \in \overline{1, B+1} \\ L_{0,0}^s(z) = \frac{\mu}{s + \mu} E^{B+1}(z) + \frac{\mu}{s + \mu} \sum_{r=1}^{B+1} a_r L_{r,0}^s(z), \quad (52)$$

where

$$\tilde{V}^k(x, z, s) = \mathbf{E} [e^{-s\chi_{r-1}^B(x)} z^T; \mathfrak{A}^k] = \tilde{f}^k(x, z, s) - \frac{Q_k^s(x)}{Q_{B+1}^s} \tilde{f}^{B+1}(0, z, s). \quad (53)$$

The equality (9) implies for the function  $\tilde{f}^k(x, z, s) = \mathbf{E} [e^{-s\tau^k(x)} z^{T^k(x)}; \mathfrak{B}^k(x)]$  that

$$\tilde{f}^k(x, z, s) = \frac{\mu}{s} \sum_{i=0}^k E^i(z) [A_x^{k-i}(s) - A_x^{k-i-1}(s)] + Q_k^s(x) F(c(s), z), \quad (54)$$

where  $F(c(s), z) = \frac{1-c(s)}{1-c(s)/z} \frac{k(z)-k(c(s))}{s-k(c(s))}$ . Solving the system (52) and taking into account (53) for all  $r \in \overline{0, B+1}$ ,  $x \geq 0$ , we get

$$L_{r,x}^s(z) = \tilde{f}^k(x, z, s) + \mu \frac{Q_k^s(x)}{Q_{B+1}^s} \frac{E^{B+1}(z) + \sum_{i=1}^{B+1} a_i \tilde{f}^{B+1-i}(0, z, s) - \tilde{f}^{B+1}(0, z, s)}{s + \mu - \mu \tilde{Q}(s)/Q_{B+1}^s}.$$

The first formula of the corollary follows from the latter equality and from (54). Comparing the coefficients of  $z^n$ ,  $n \in \mathbb{N}$  in both sides of (51), we derive the second formula of the corollary.  $\square$

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